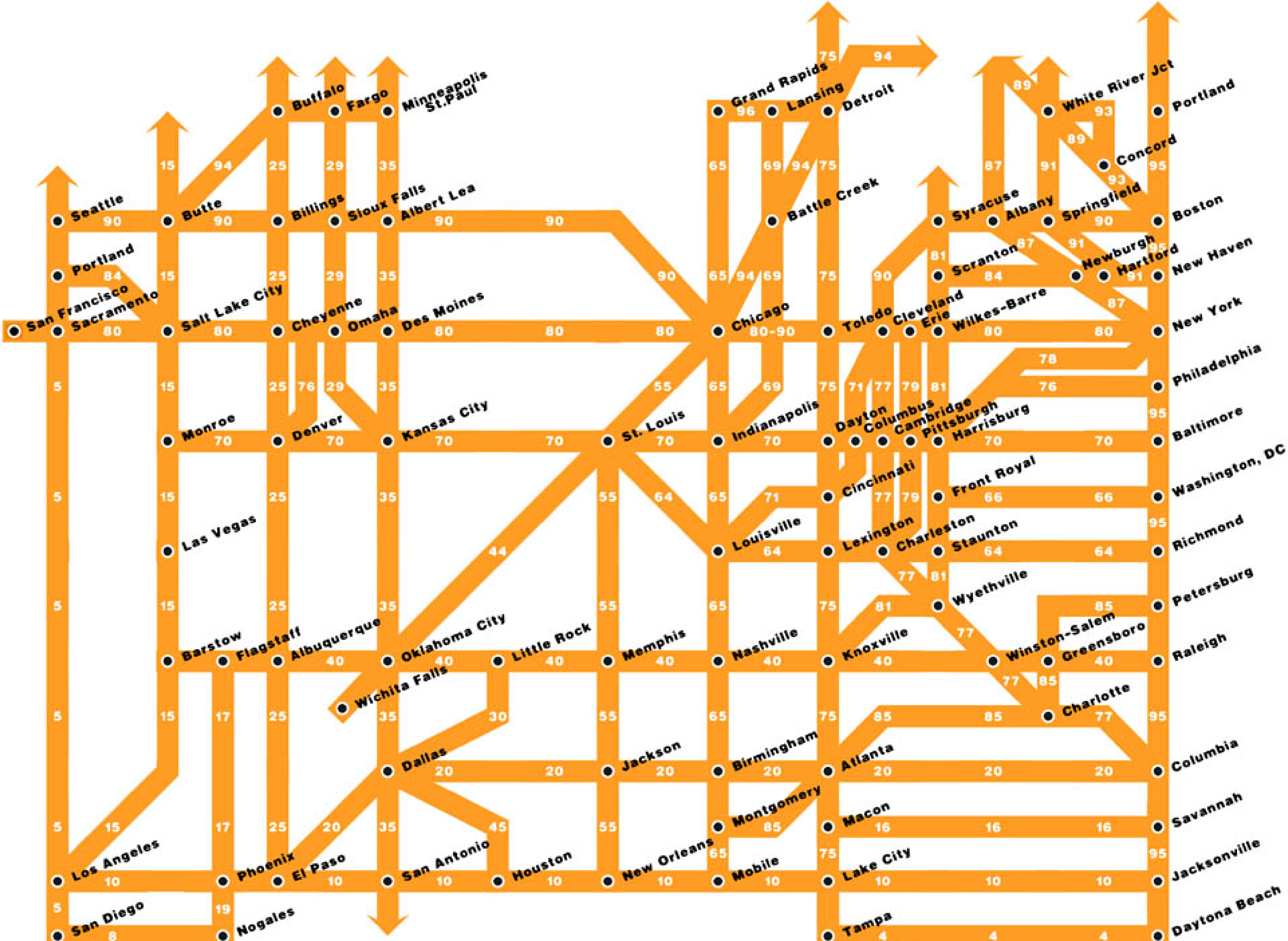
Graph Theory

Part One

Outline for Today

- ***Graphs and Digraphs***
  - Two fundamental mathematical structures.
- ***Independent Sets and Vertex Covers***
  - Two structures in graphs.
- ***Proofs on Graphs***
  - Reprising themes from last week.

Graphs and Digraphs



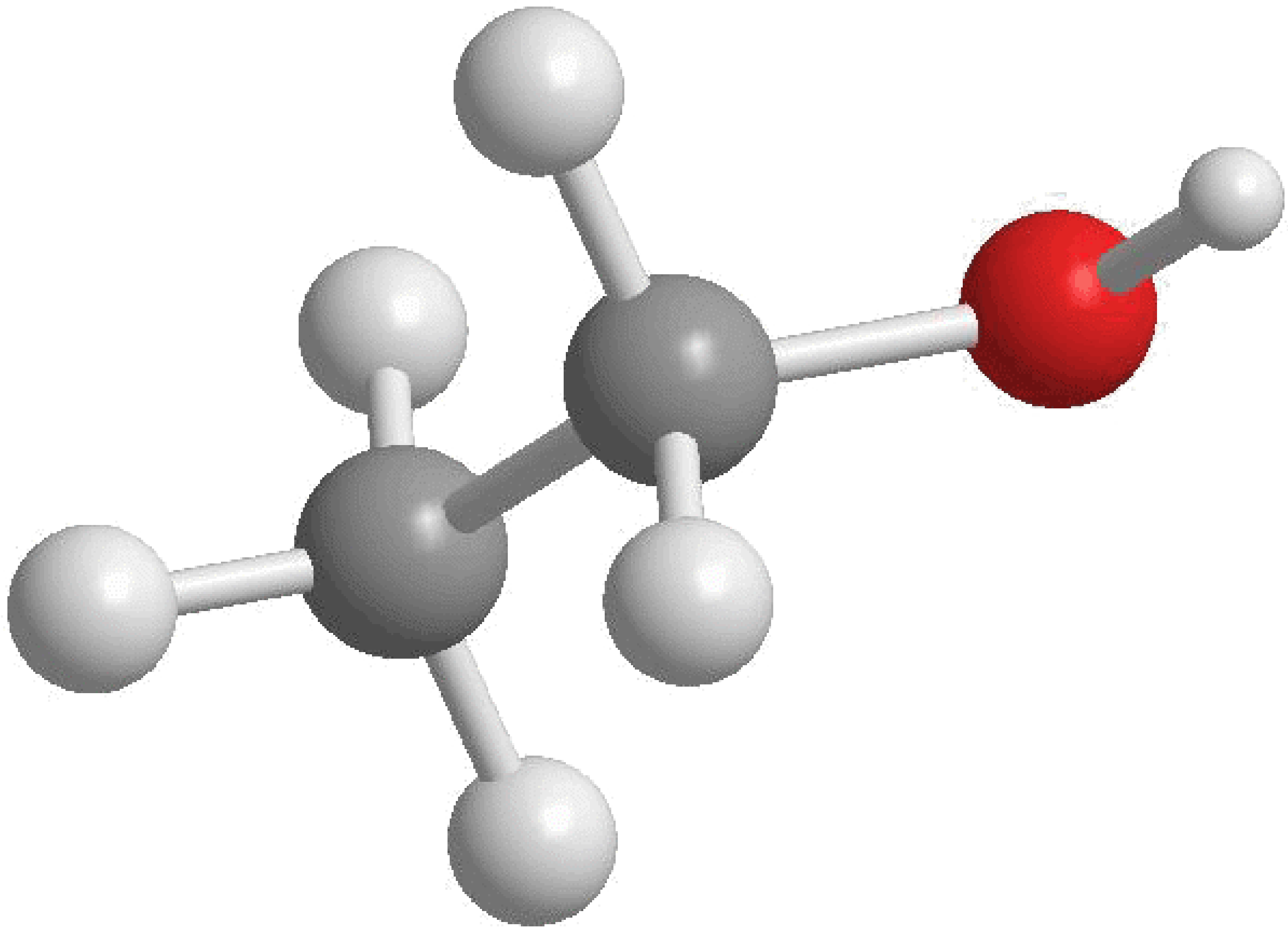
THE EISENHOWER INTERSTATE SYSTEM

(simplified)

CHRIS YATES 2007

http://strangemaps.files.wordpress.com/2007/02/fullinterstatemap-web.jpg

Chemical Bonds



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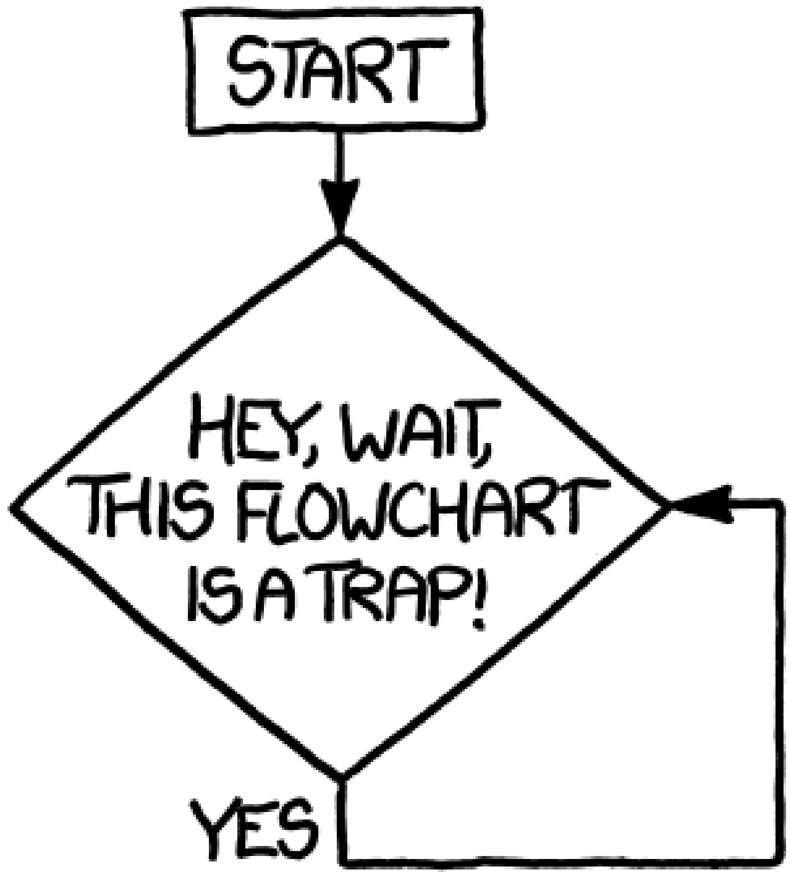
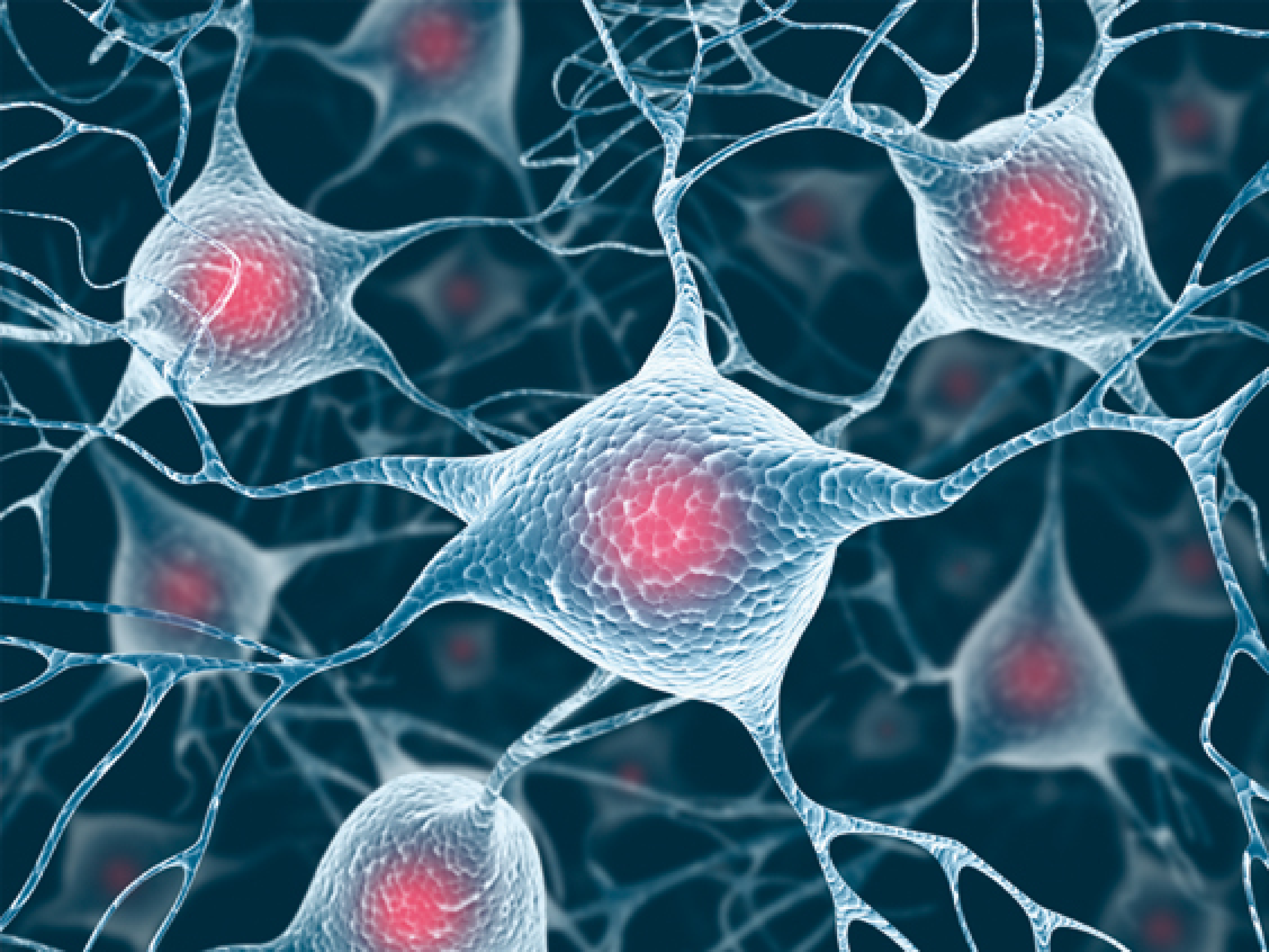

START
HEY, WAIT, THIS FLOWCHART IS A TRAP!
YES



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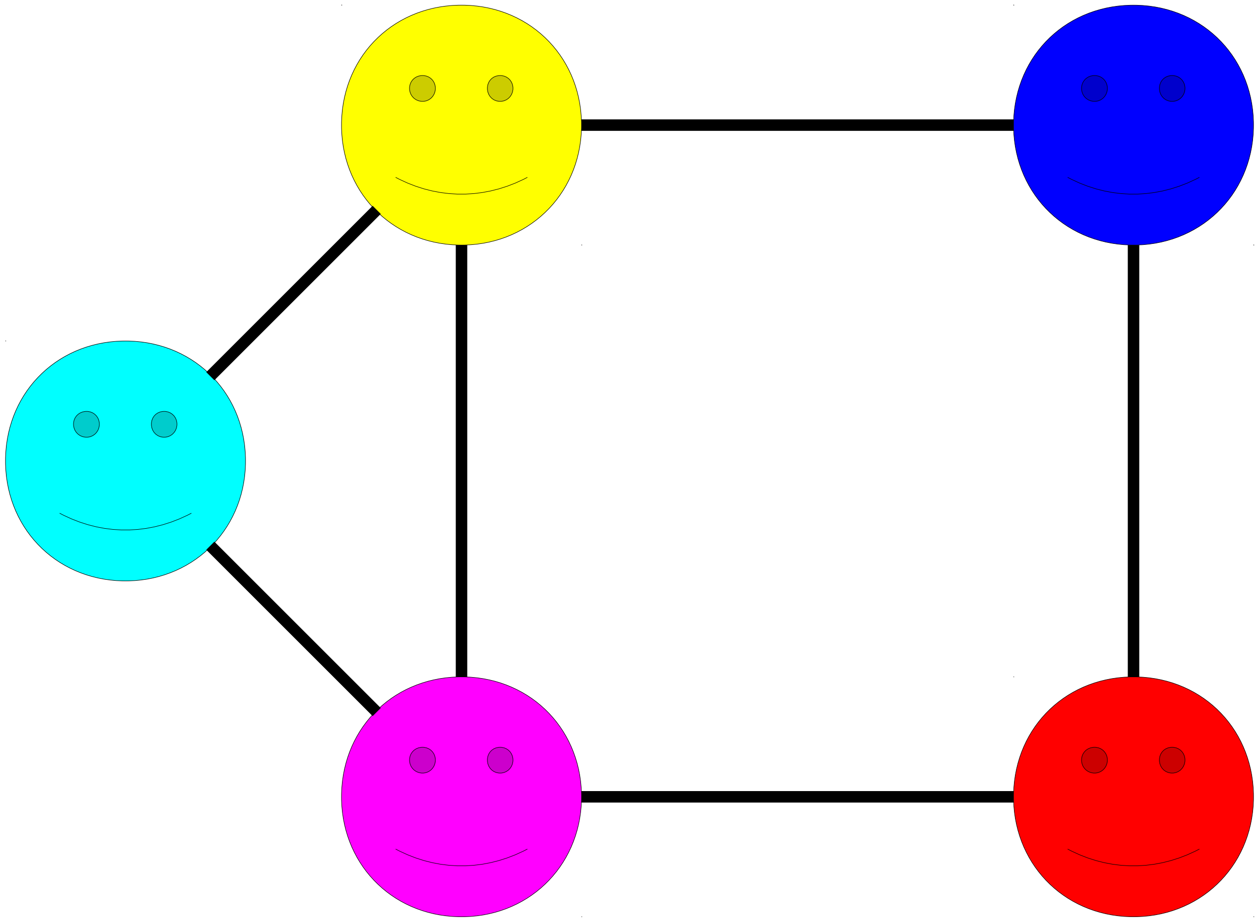
What's in Common

- Each of these structures consists of
  - a collection of objects and
  - links between those objects.
- ***Goal:*** find a general framework for describing these objects and their properties.

A ***graph*** is a mathematical structure for representing relationships.



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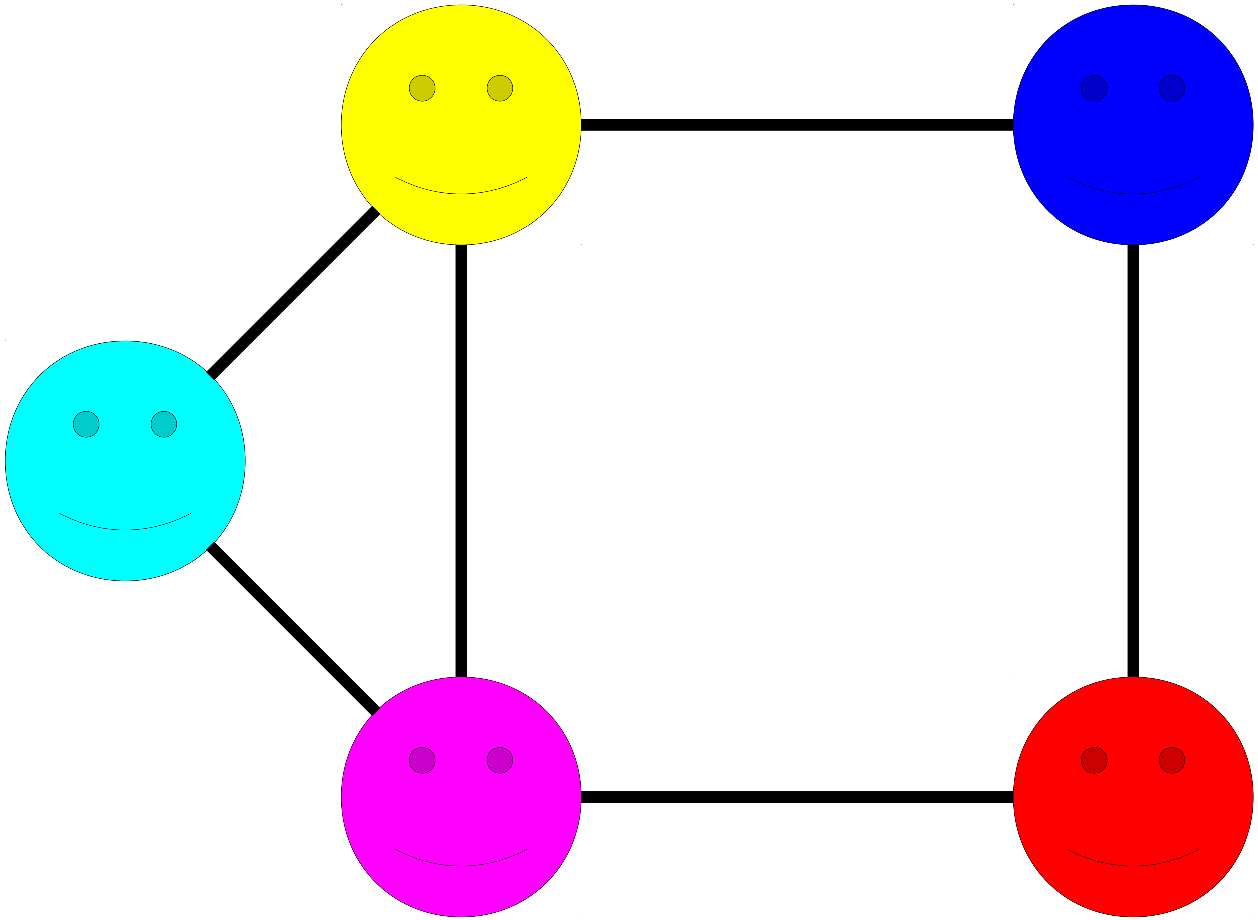

A graph consists of a set of ***nodes*** (or ***vertices***) connected by ***edges*** (or ***arcs***)

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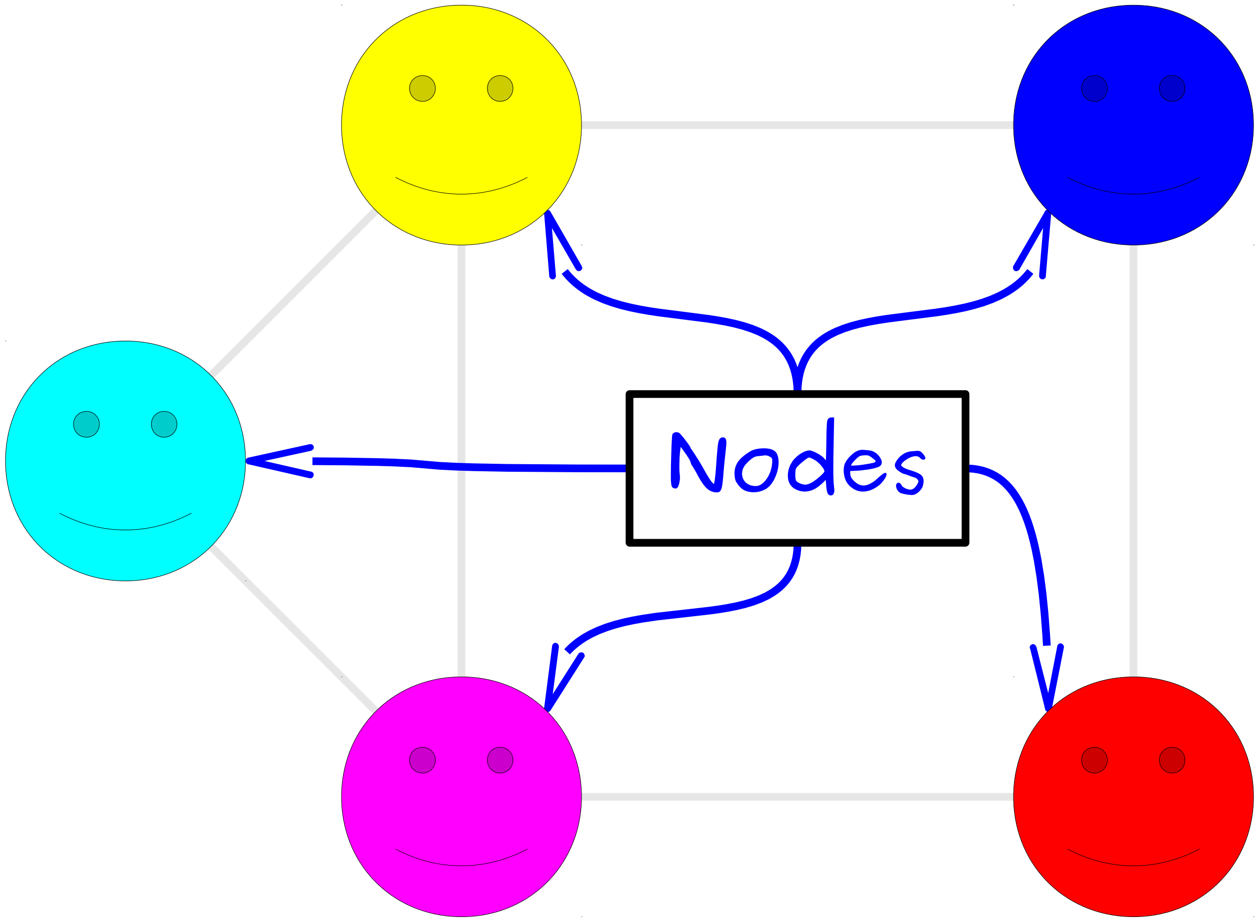

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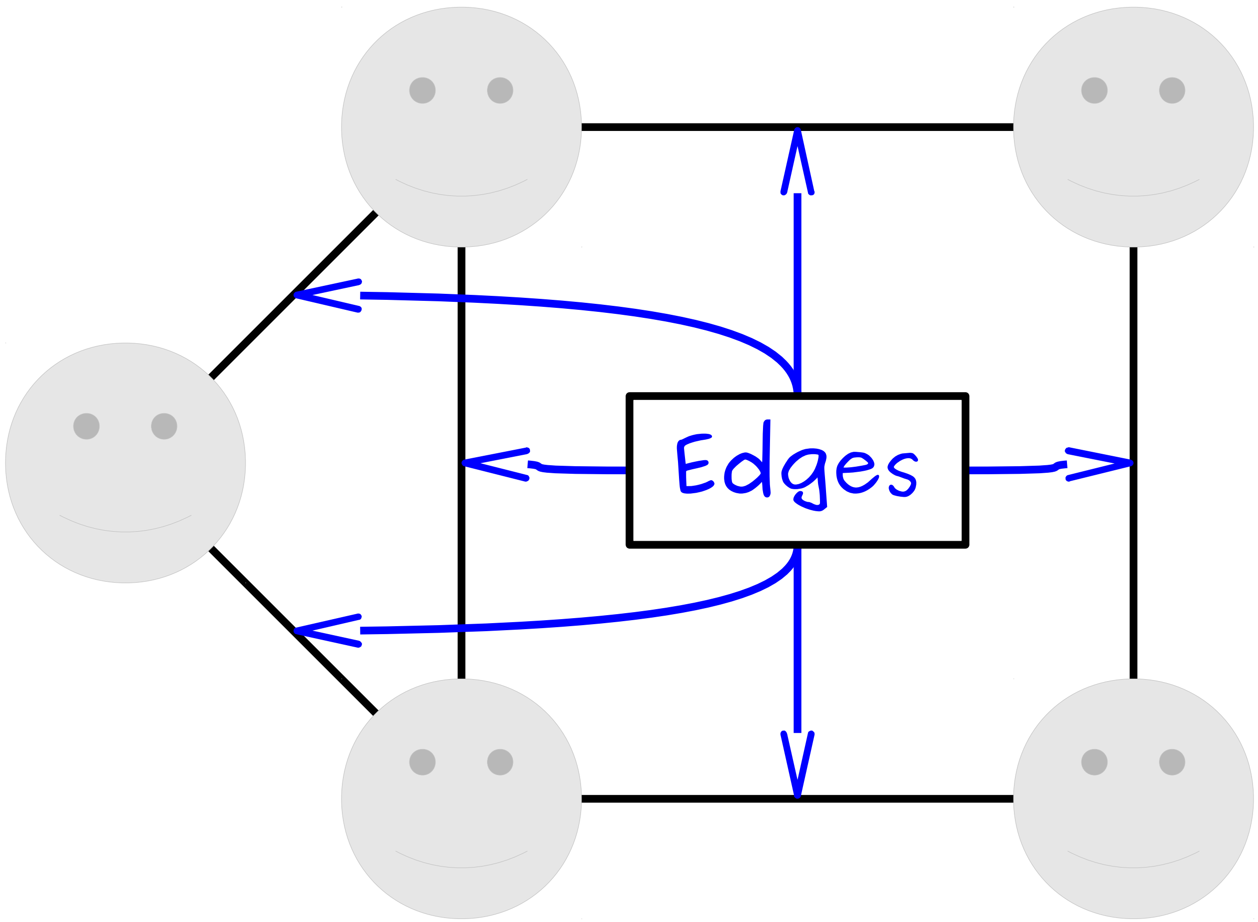

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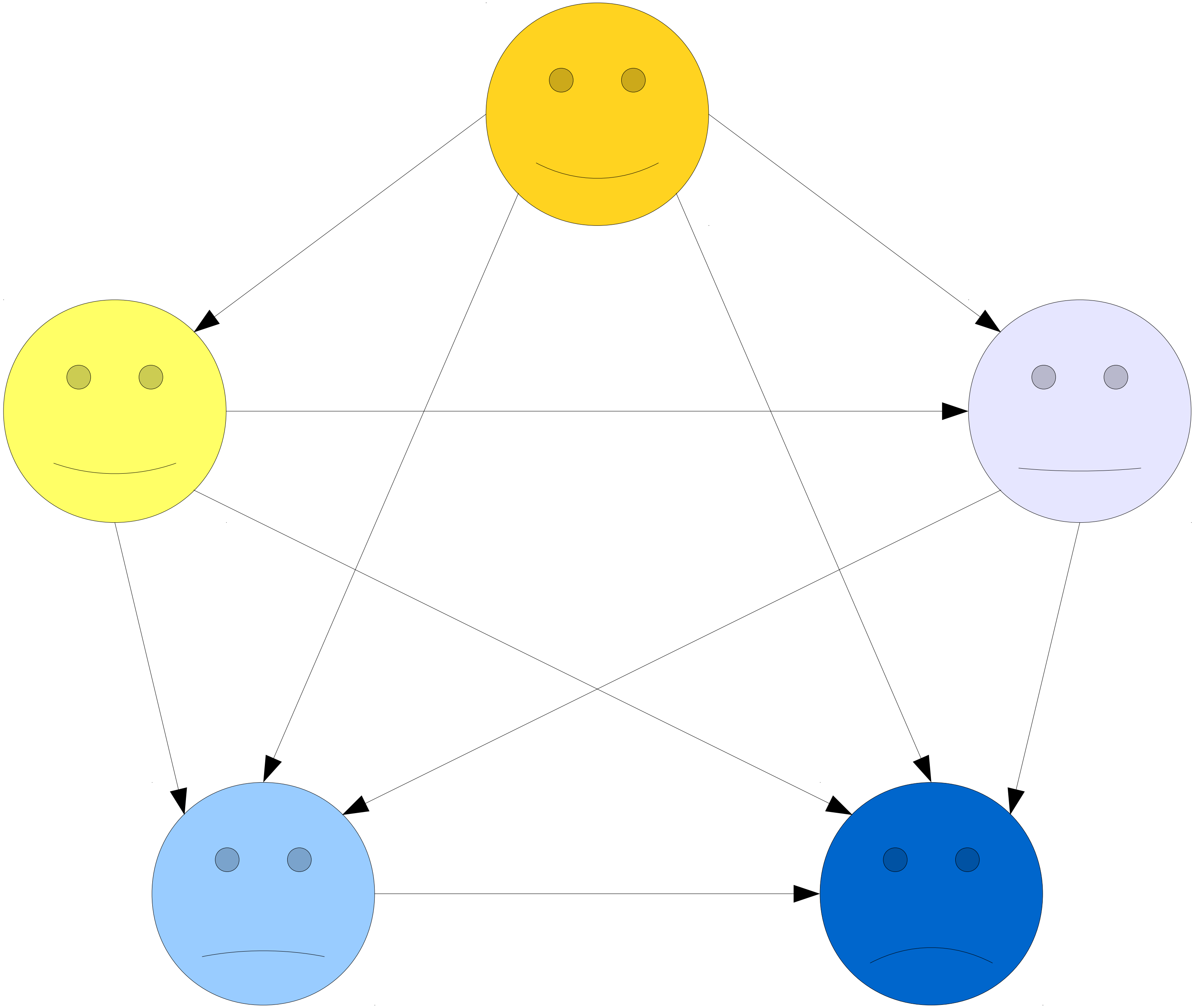
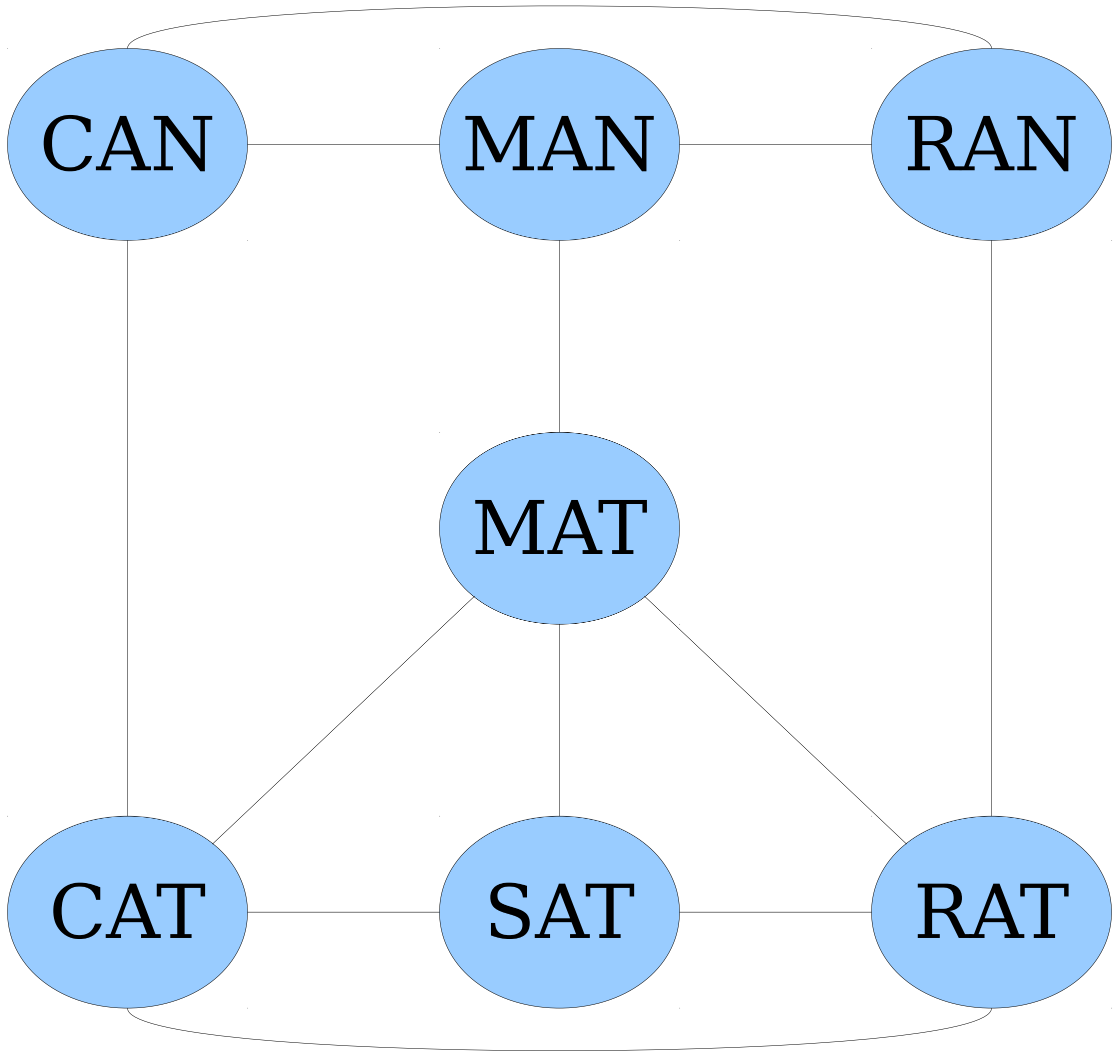
Some graphs are ***directed***.



Some graphs are ***undirected***.



Graphs and Digraphs

- An ***undirected graph*** is one where edges link nodes, with no endpoint preferred over the other.
- A ***directed graph*** (or ***digraph***) is one where edges have an associated direction.
  - (There's something called a ***mixed graph*** that allows for both types of edges, but they're fairly uncommon and we won't talk about them.)
- Unless specified otherwise:

☞ ***"Graph" means "undirected graph"*** ☜

Formalizing Graphs

- How might we define a graph mathematically?
- We need to specify
  - what the nodes in the graph are, and
  - which edges are in the graph.
- The nodes can be pretty much anything.
- What about the edges?

Formalizing Graphs

- An ***unordered pair*** is a set $\{a, b\}$ of two elements $a \neq b$. (Remember that sets are unordered.)
  - For example, $\{0, 1\} = \{1, 0\}$
- An ***undirected graph*** is an ordered pair $G = (V, E)$, where
  - V is a set of nodes, which can be anything, and
  - E is a set of edges, which are *unordered* pairs of nodes drawn from V.
- A ***directed graph*** (or ***digraph***) is an ordered pair $G = (V, E)$, where
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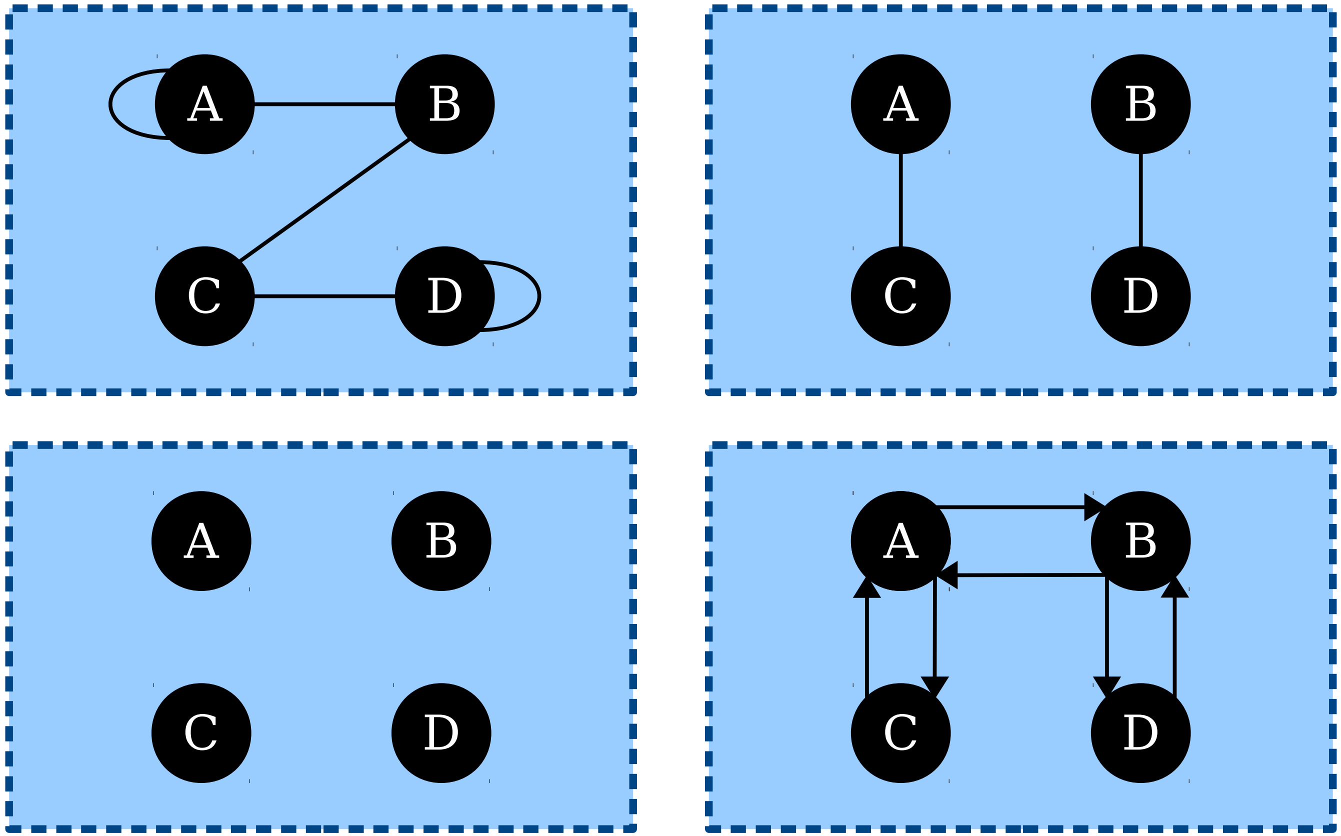
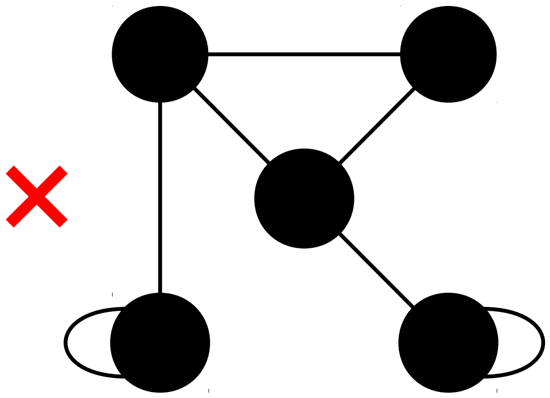


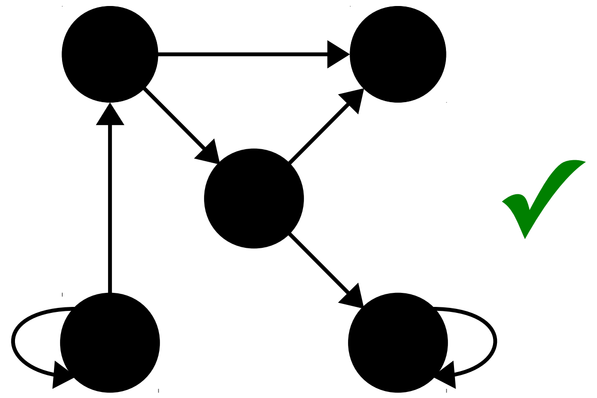
How many of these drawings are of valid undirected graphs?

Answer at
https://pollev.com/cs103

Self-Loops

- An edge from a node to itself is called a ***self-loop***.
- In (undirected) graphs, self-loops are generally not allowed.
  - Can you see how this follows from the definition?
- In digraphs, self-loops are generally allowed unless specified otherwise.





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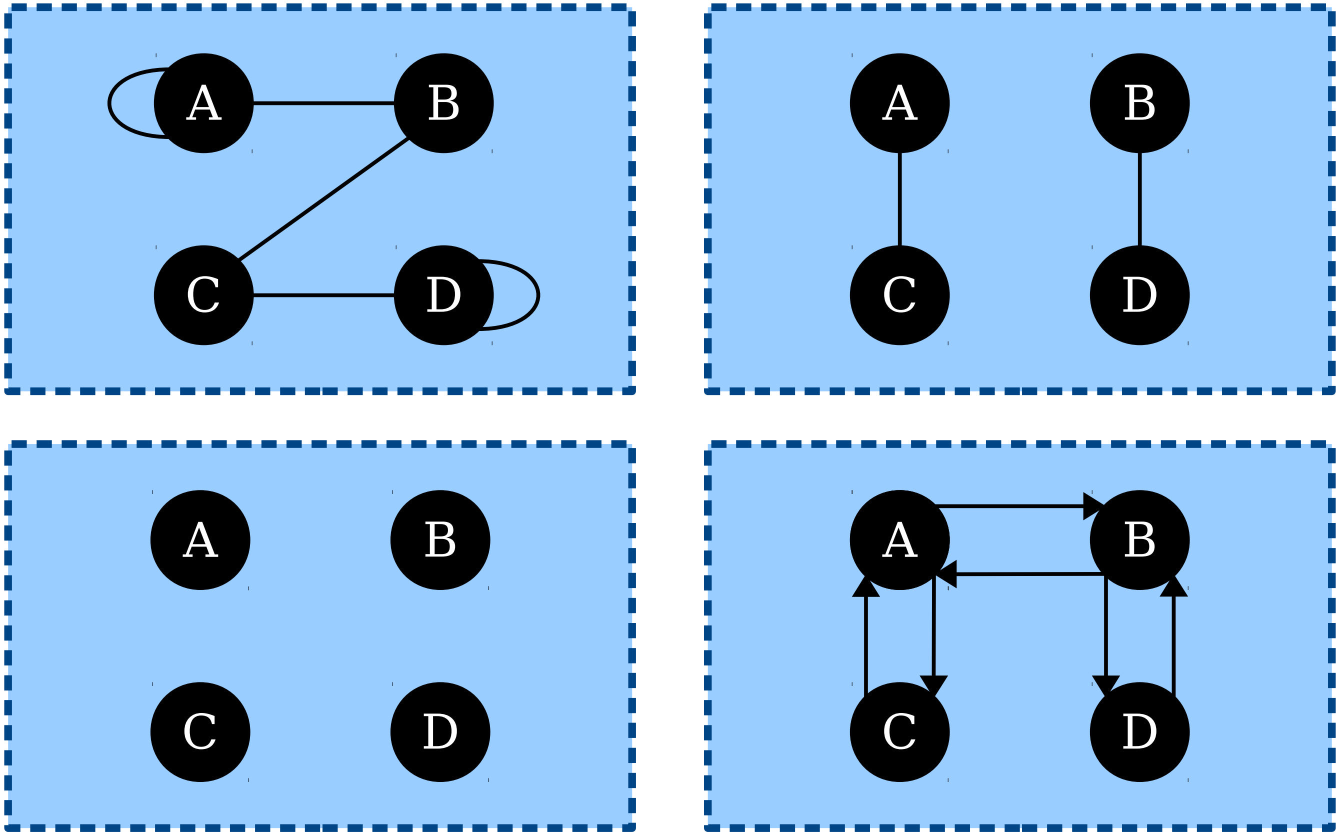

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Time-Out for Announcements!

PS2 Solutions Released

- Solutions to Problem Set Two are now available on the course website.
  - We generally don't release solutions to autograded problems.
  - If you have any questions about those, ping us privately over EdStem or come talk to us at our office hours.
- PS3 is due this Friday at 4:00PM. Ask questions if you have them! That's what we're here for.

Midterm Exam Logistics

- Our first midterm exam is next ***Tuesday, February 6th***, from ***7:00PM – 10:00PM***. Locations are divvied up by last (family) name:
  - A – P: Go to Hewlett 200.
  - Q – Z: Go to Hewlett 201.
- You're responsible for Lectures 00 – 05 and topics covered in PS1 – PS2. Later lectures (functions forward) and problem sets (PS3 onward) won't be tested here. Exam problems may build on the written or coding components from the problem sets.
- The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5" × 11" sheet of notes with you to the exam, decorated however you'd like.
- Students with OAE accommodations should have received an email with exam logistics. Please double check this ***TODAY***.

Midterm Exam

- ***We want you to do well on this exam.***
  - We're not trying to “weed out” weak students.
  - We're not trying to enforce a curve where there isn't one.
  - We want you to show what you've learned up to this point so that you get a sense for where you stand and where you can improve.
- The purpose of this midterm is to give you a chance to show what you've learned in the past few weeks.
- It is not designed to assess your “mathematical potential” or “innate mathematical ability.”

Preparing for the Exam

Extra Practice Problems

- Up on the course website, you'll find Extra Practice Problems 1 and two practice midterms.
- ***Our Recommendation:***
  - Work through practice exam 1 under realistic conditions (block off three hours, have your notes sheet, use pencil and paper).
  - Review the solutions only when you're done. ***Don't peek!*** You can't do that on the actual exam.
  - Ping the course staff to ask questions.
  - If there are concepts you need to brush up on, try some of the extra practice problems and/or review past problems from the lectures and assignments.
  - Repeat with practice exam 2!

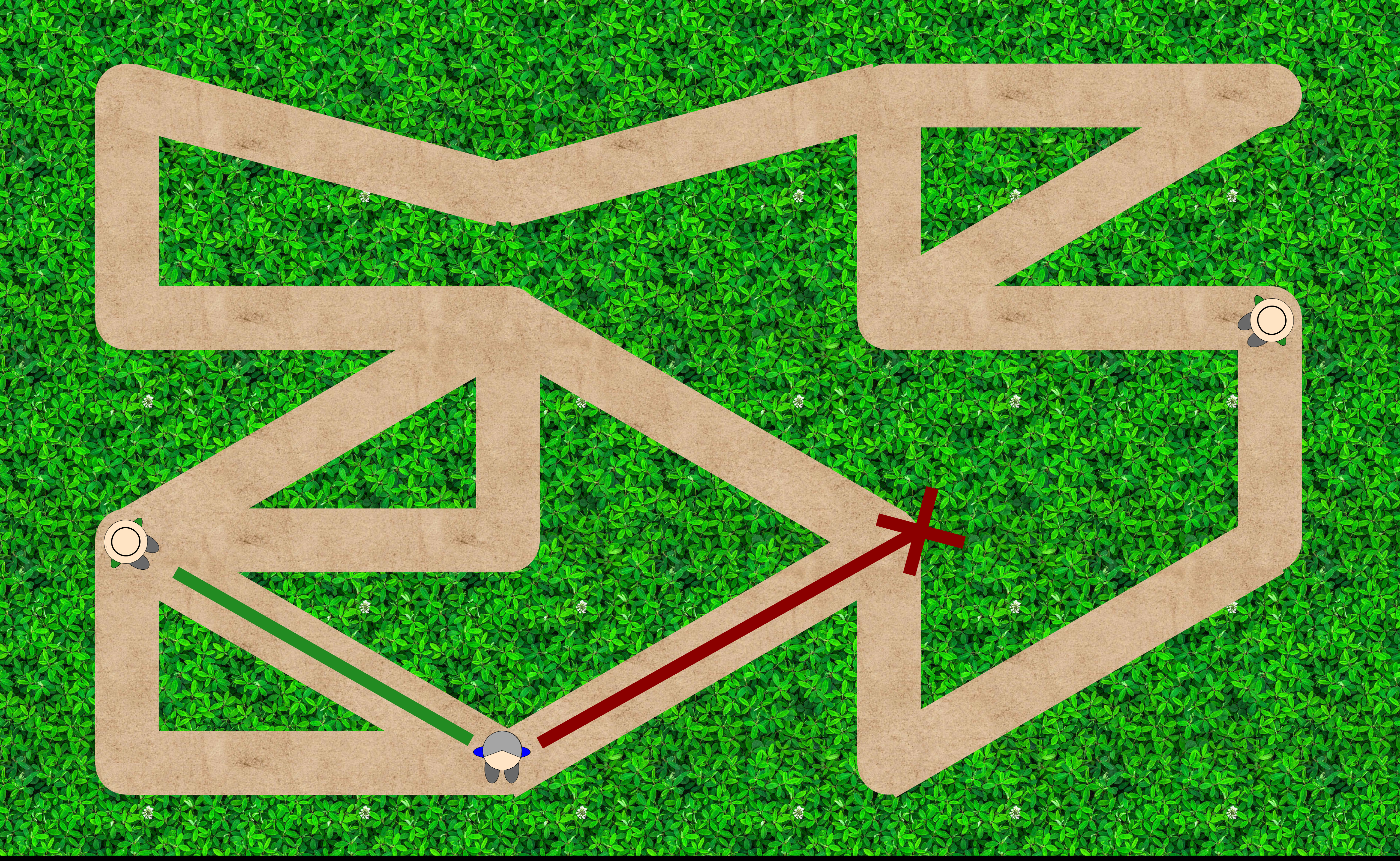
Doing Practice Problems

- As you work through practice problems, ***keep other humans in the loop!***
- Ask your problem set partner to review your answers and offer feedback – and volunteer to do the same!
- Post your answers as private questions on EdStem and ask for TA feedback! We can answer anything, from “please review this proof I wrote for one of the exam questions” or “why doesn’t the solution do *X*, which seems easier than *Y*, which is what it did?”
- ***Feedback loops are key to improving!***

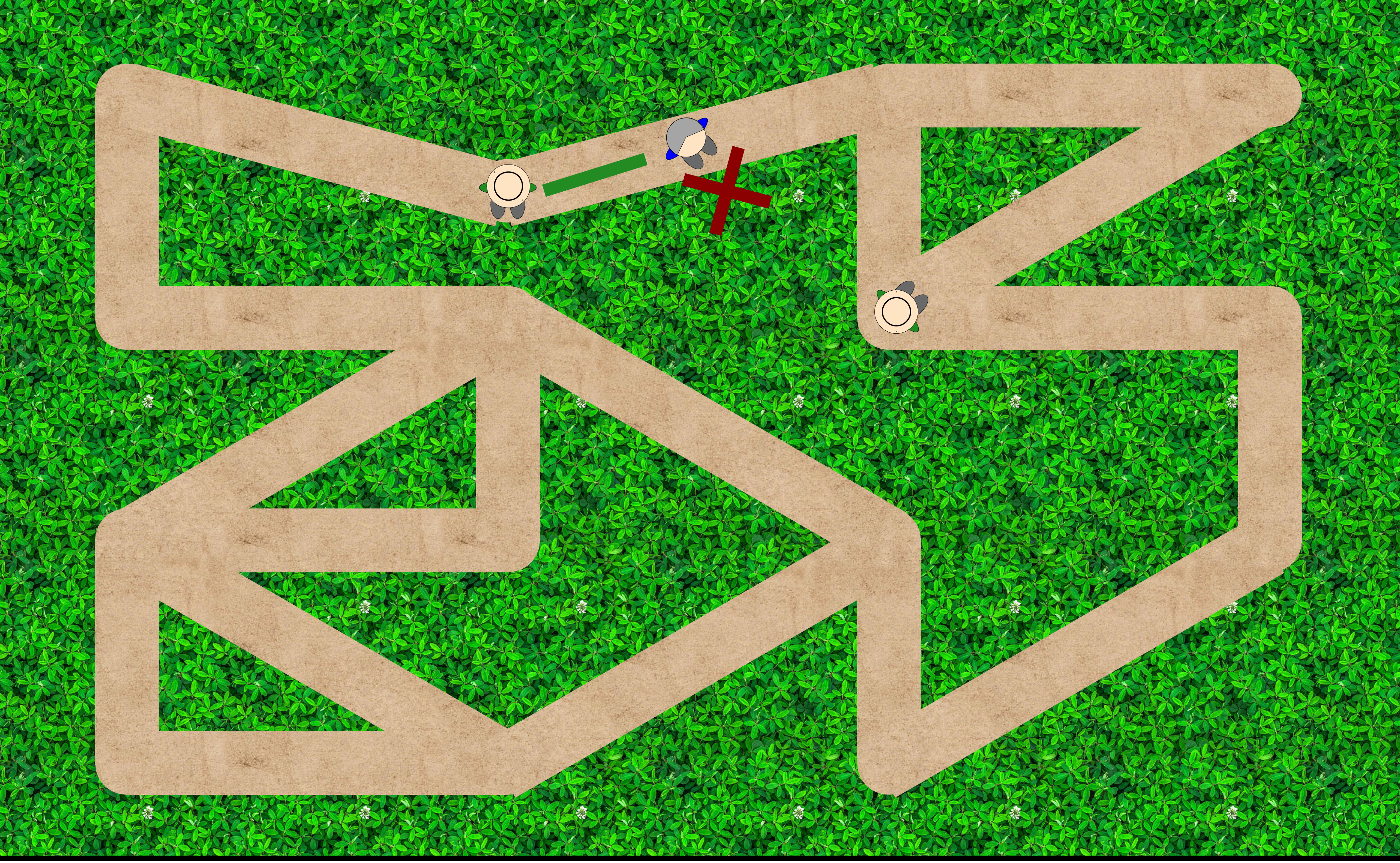
Back to CS103!

Independent Sets and Vertex Covers

Two Motivating Problems



Place park rangers in these forest trails so that
a hiker anywhere on a trail can see a park ranger.



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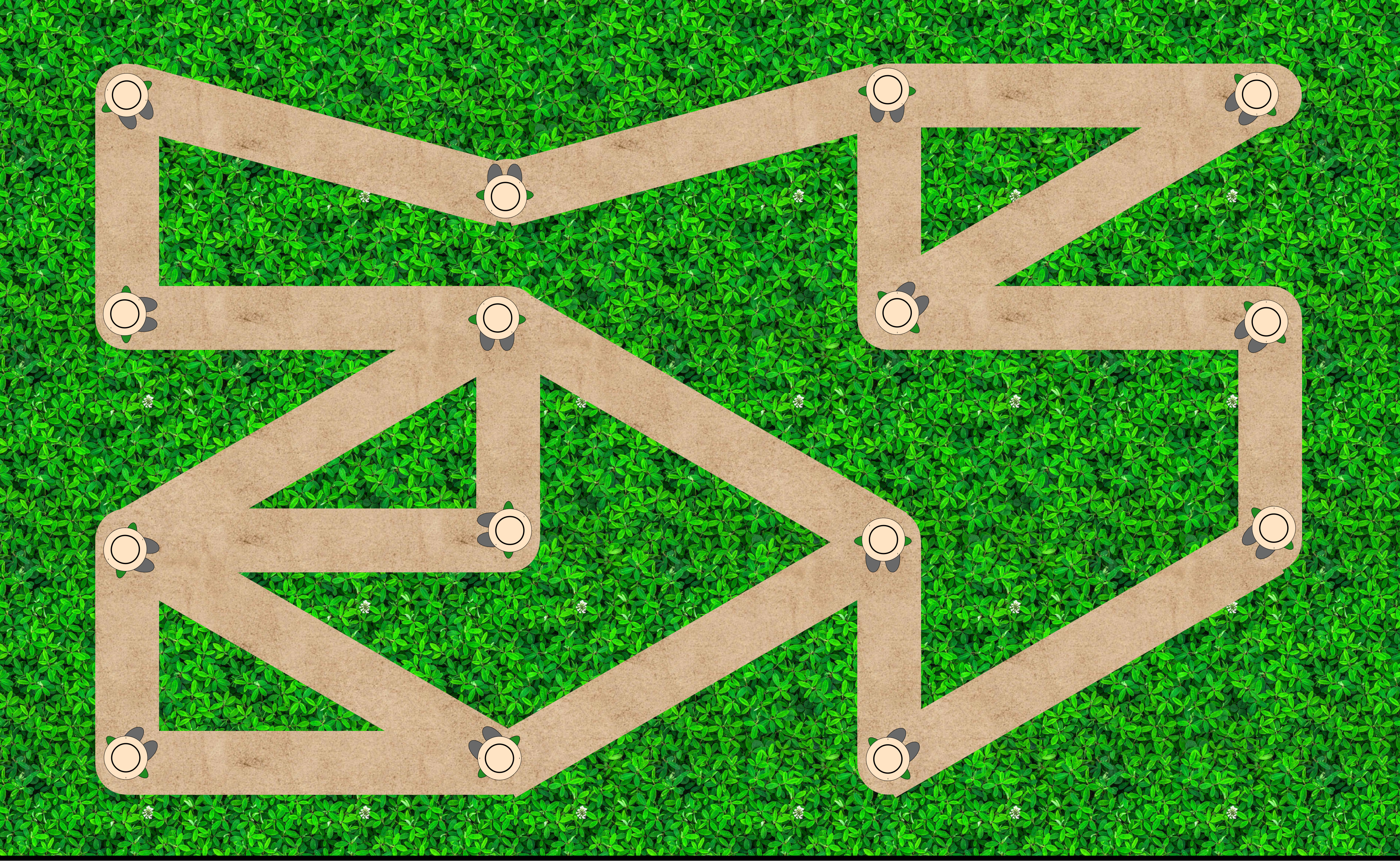

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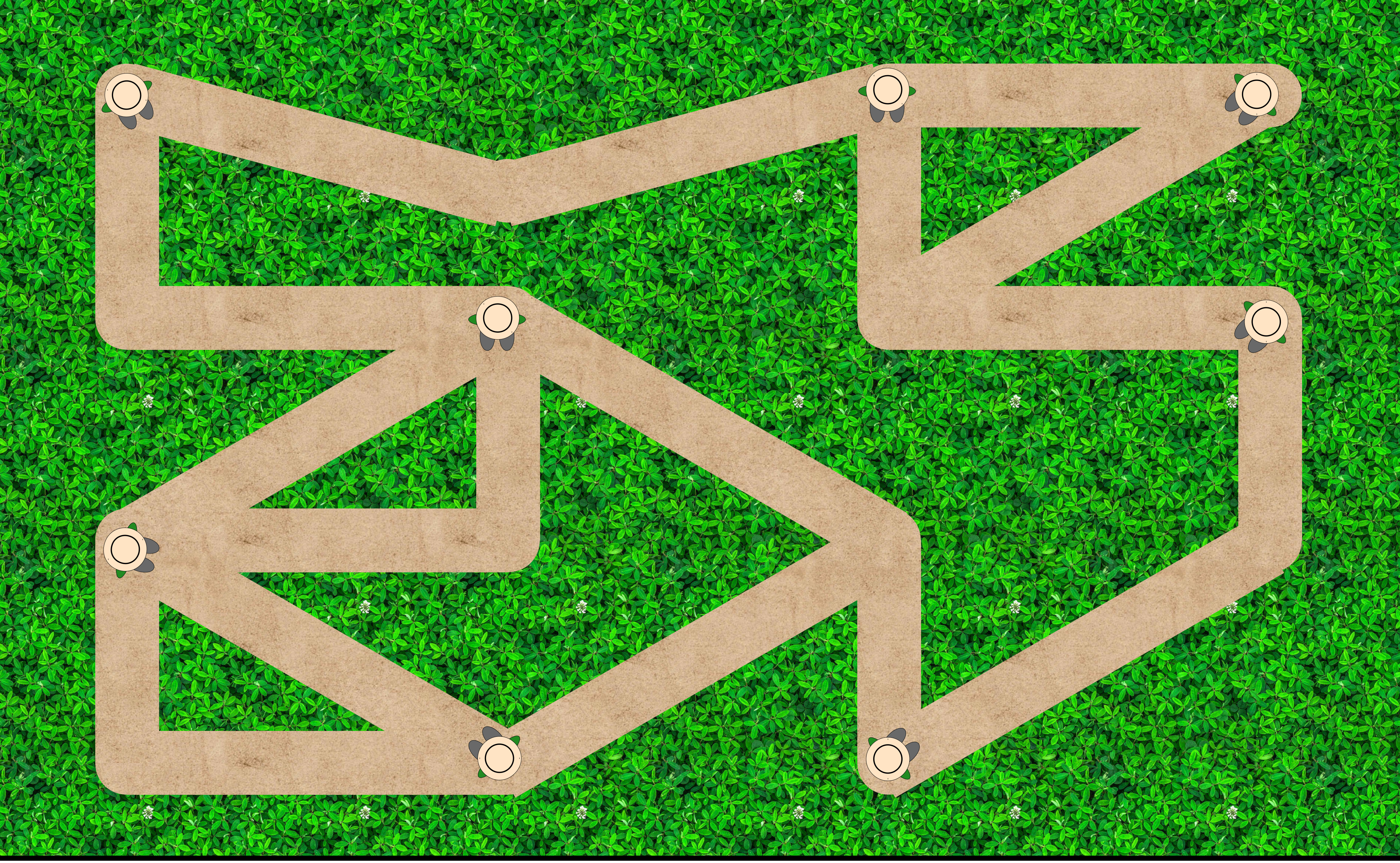

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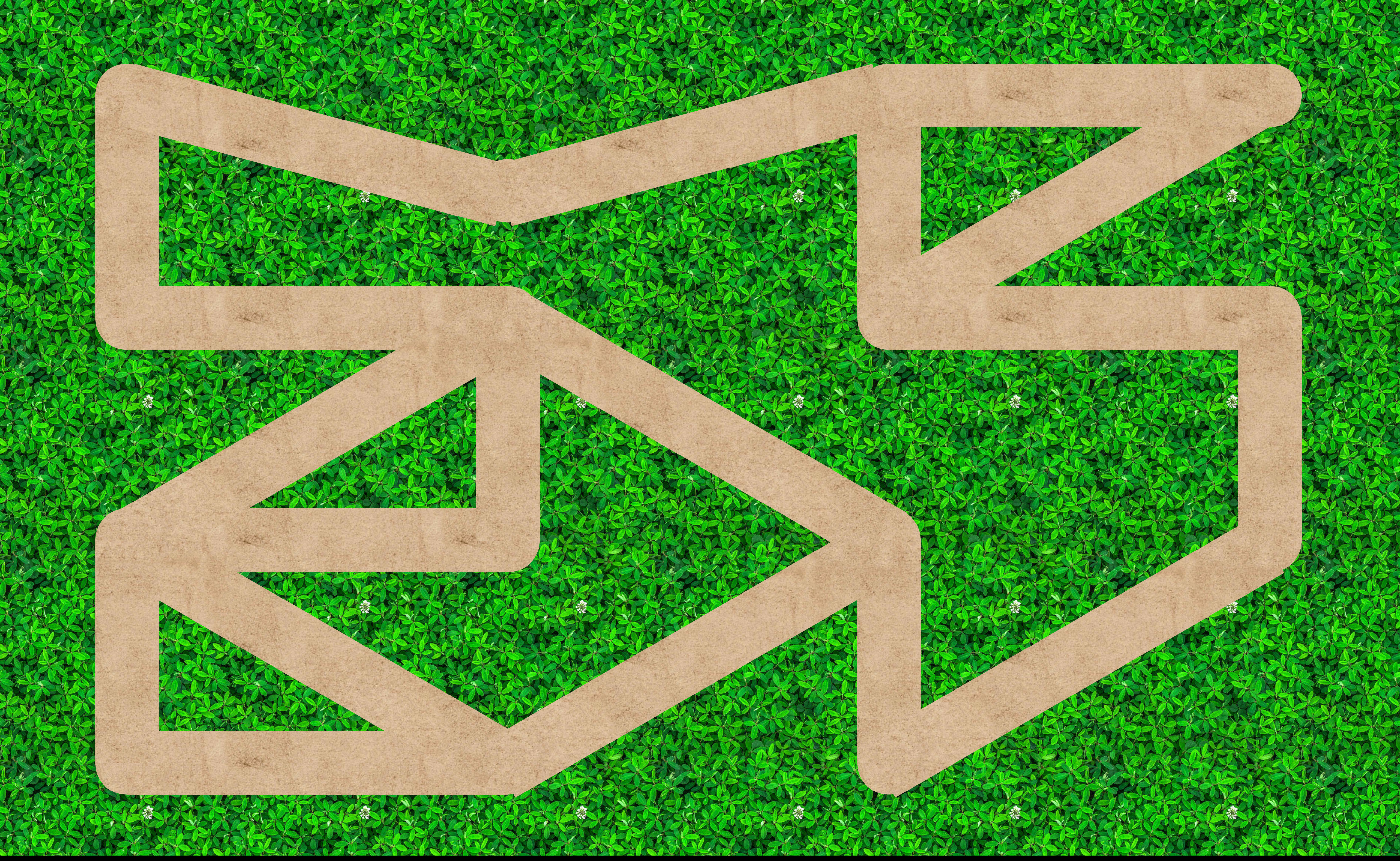

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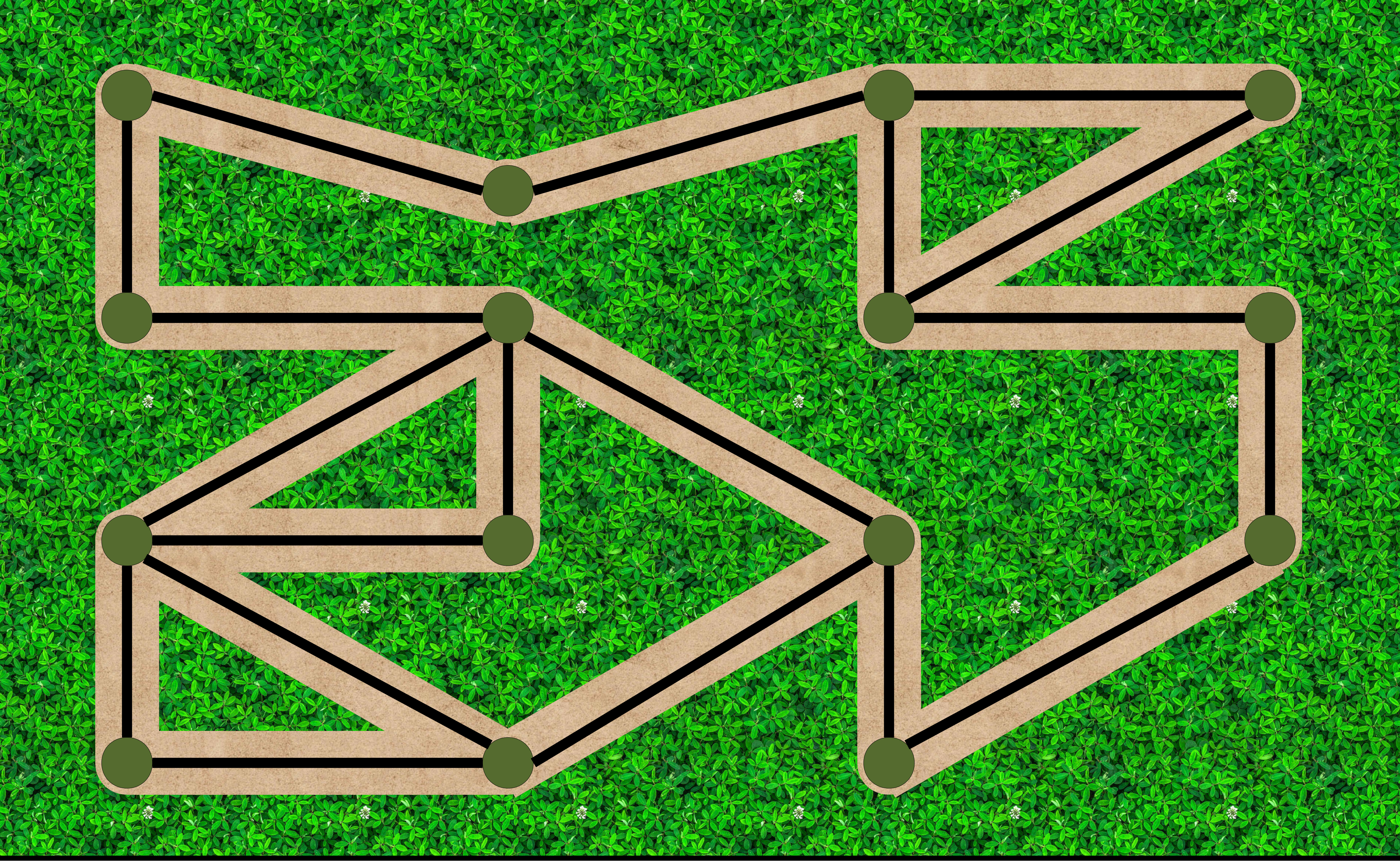

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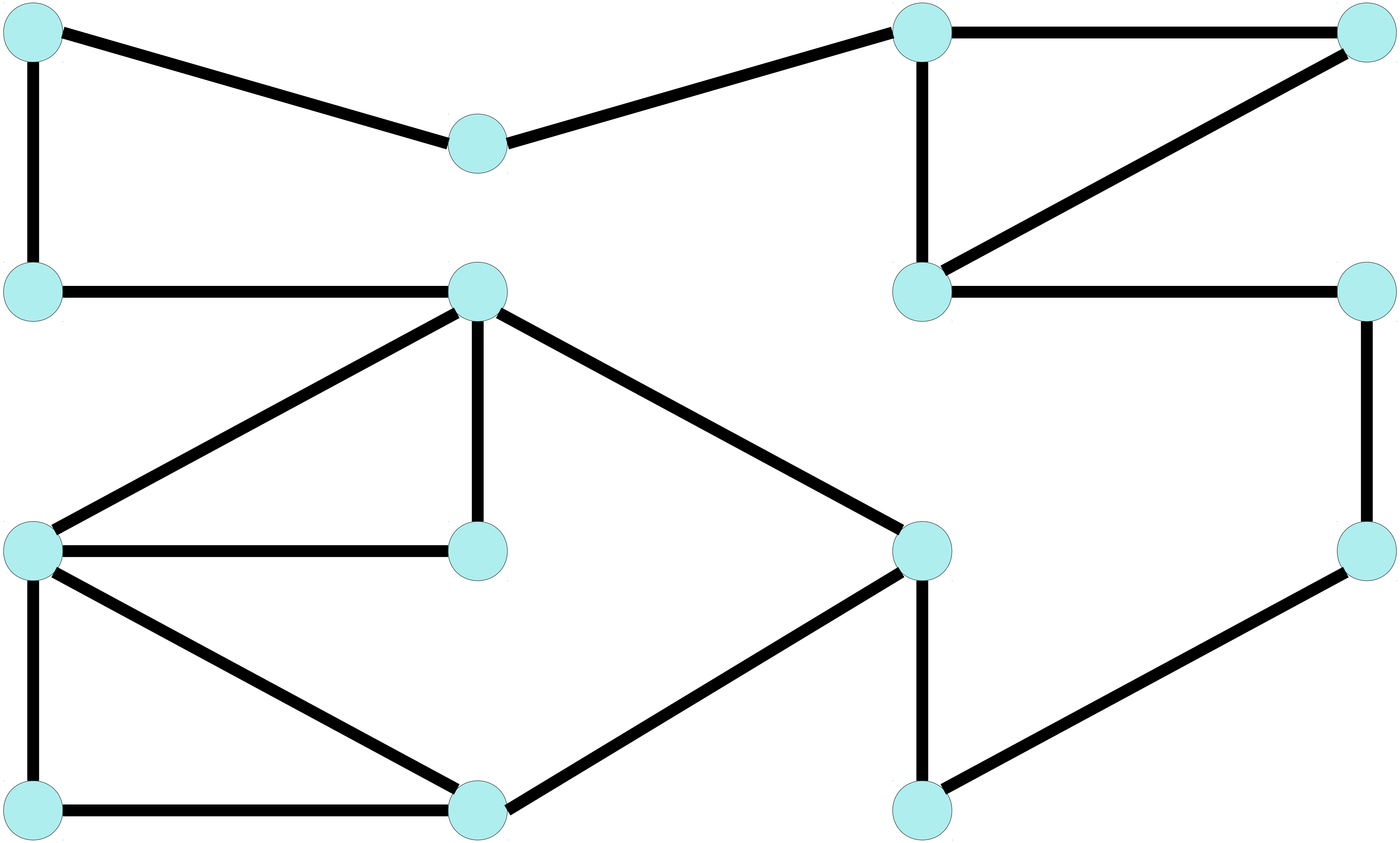

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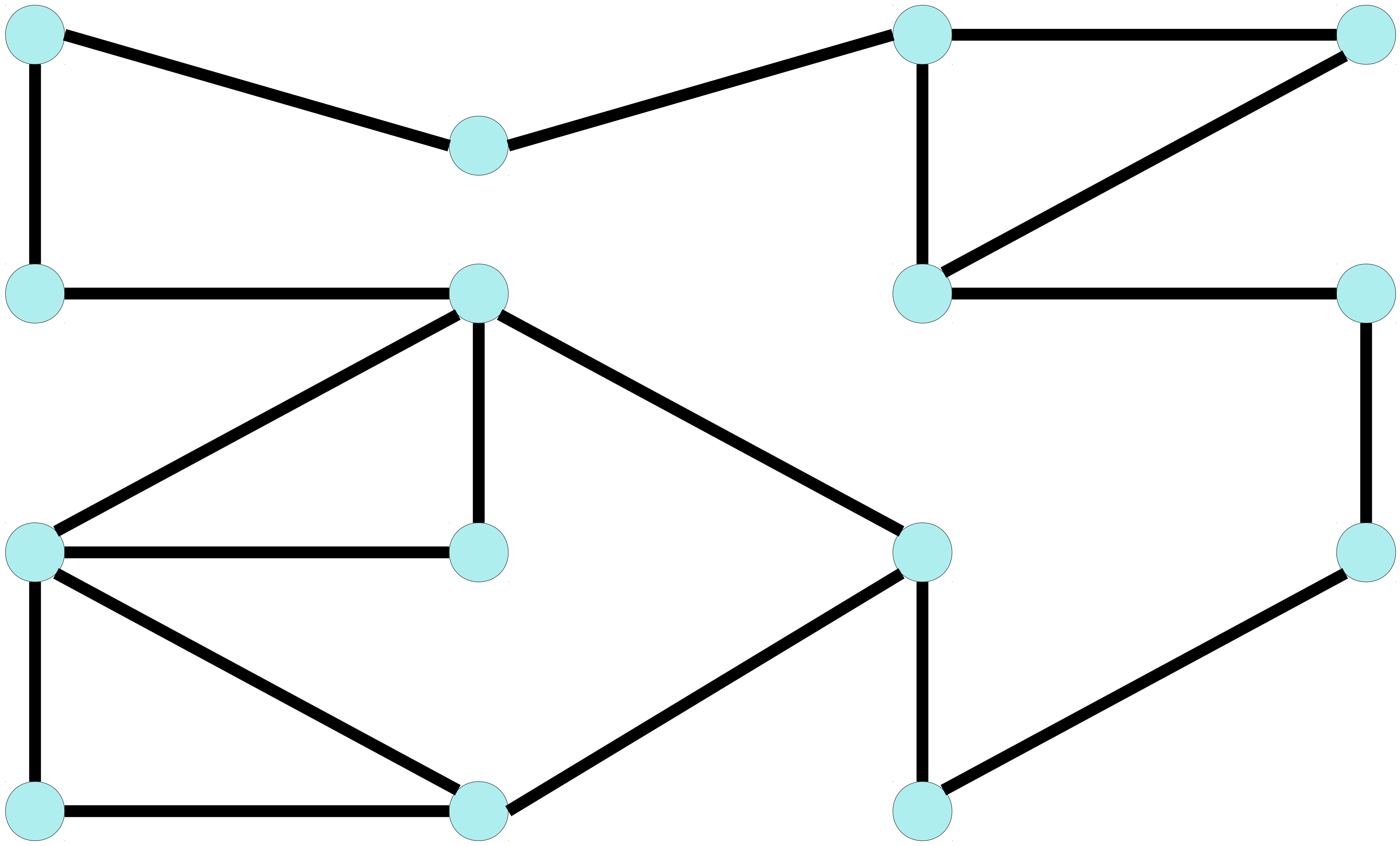
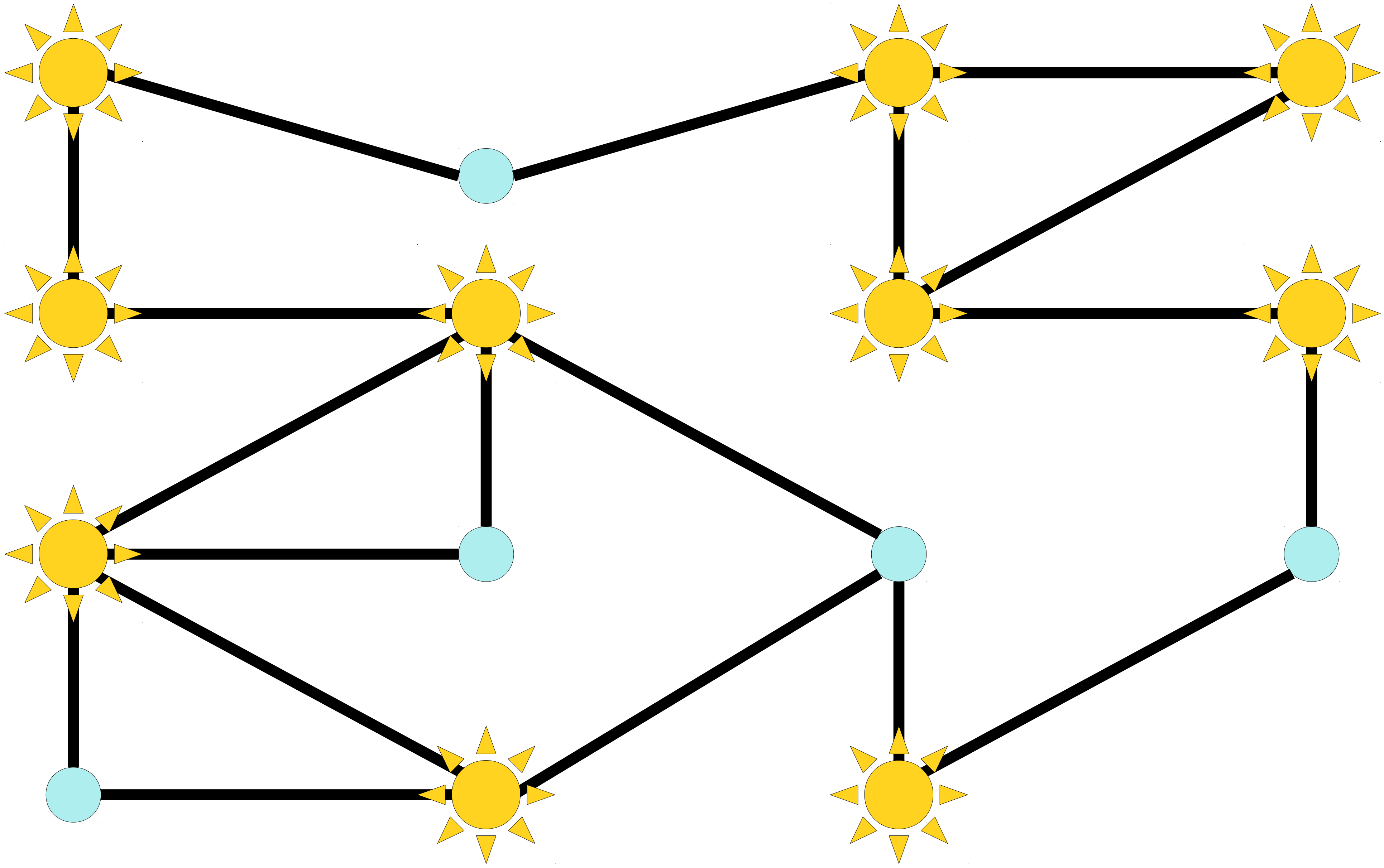


Choose at least one endpoint of each edge.



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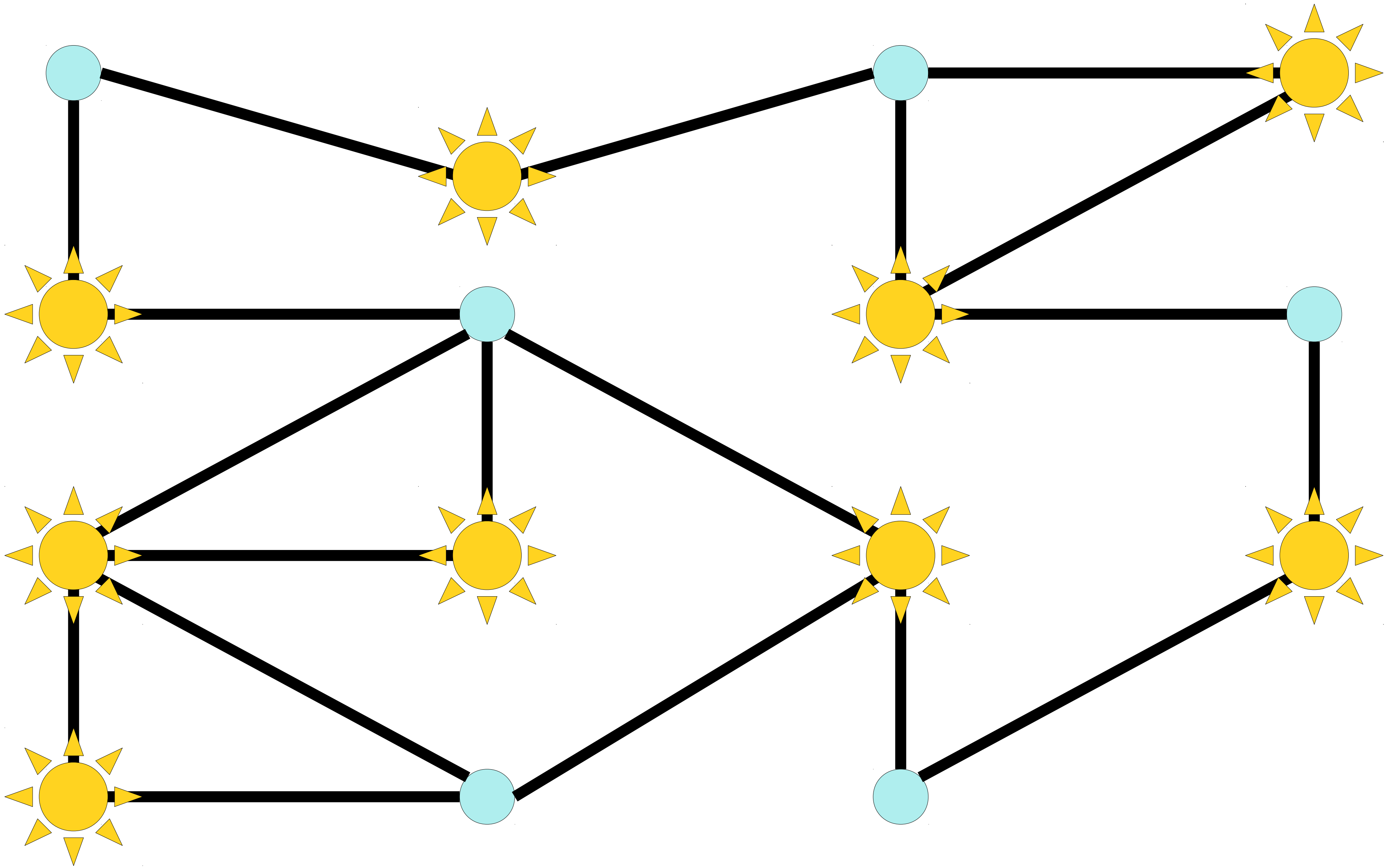

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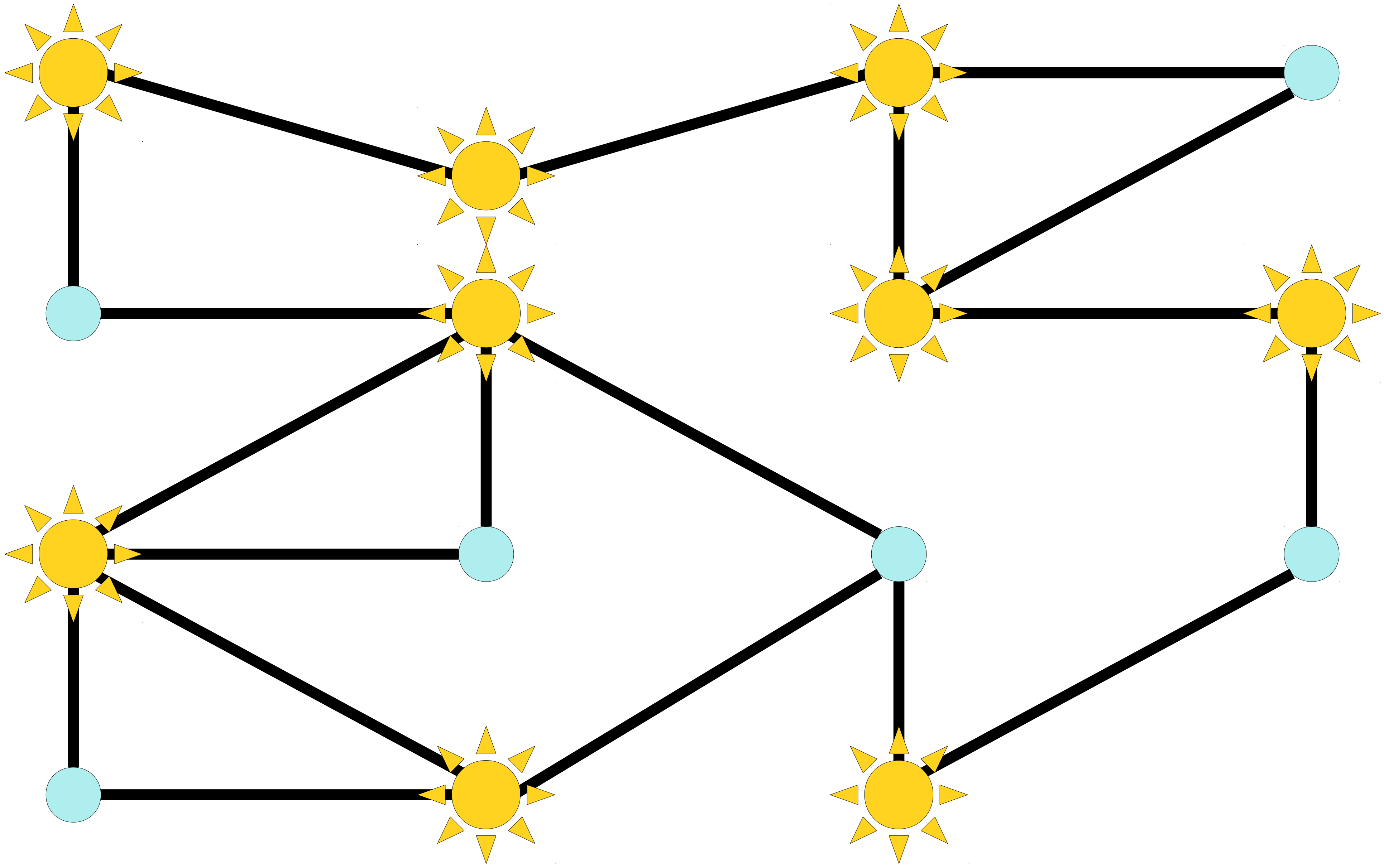


Choose at least one endpoint of each edge.

Vertex Covers

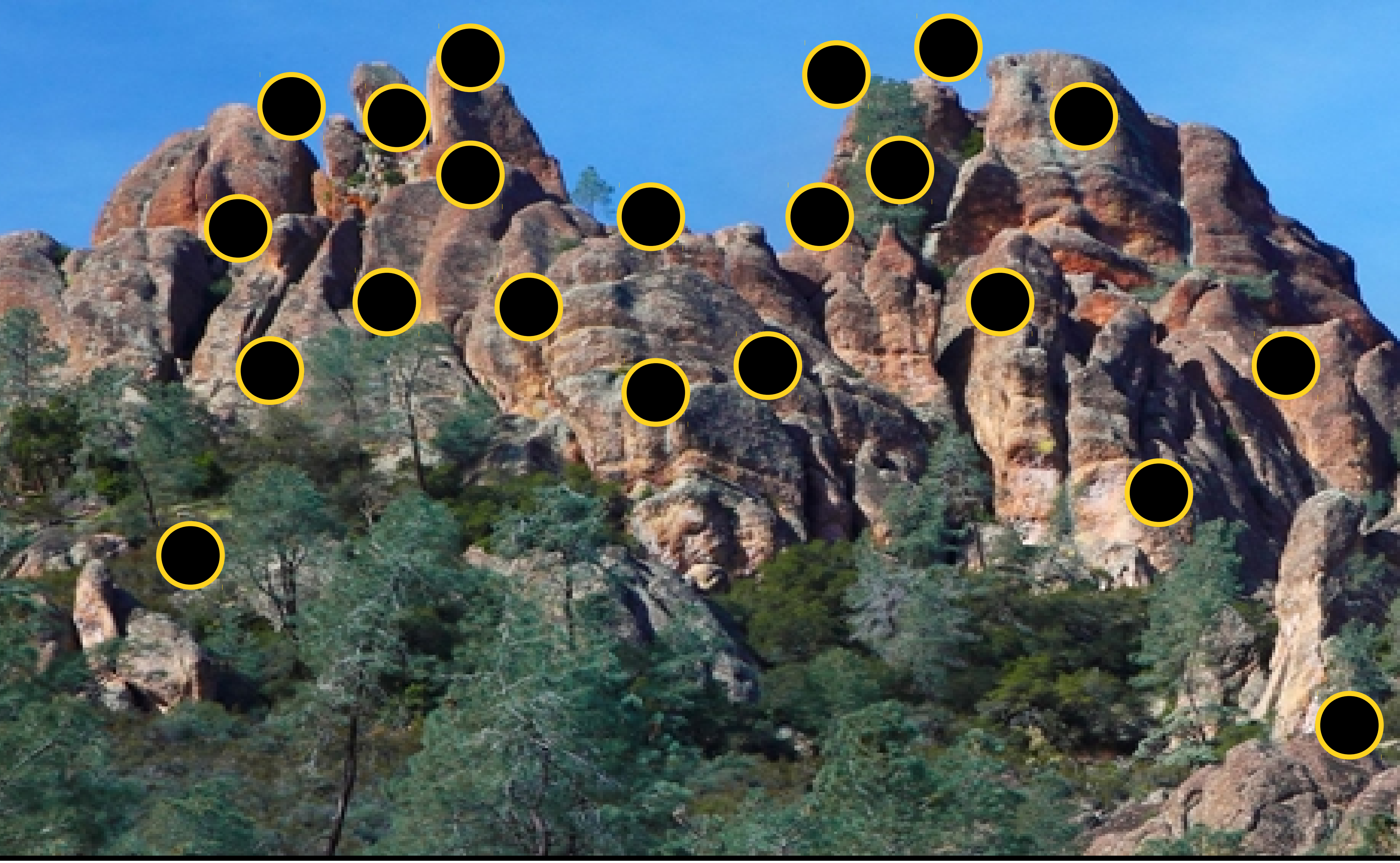
- Let $G = (V, E)$ be an undirected graph. A ***vertex cover*** of G is a set $C \subseteq V$ such that the following statement is true:

$$\forall x \in V.\ \forall y \in V.\ (\{x, y\} \in E \to (x \in C \lor y \in C))$$

("Every edge has at least one endpoint in C.")

- Intuitively speaking, a vertex cover is a set formed by picking at least one endpoint of each edge in the graph.
- Vertex covers have applications to placing streetlights/benches/security guards, as well as in gene sequencing, machine learning, and combinatorics.

Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.



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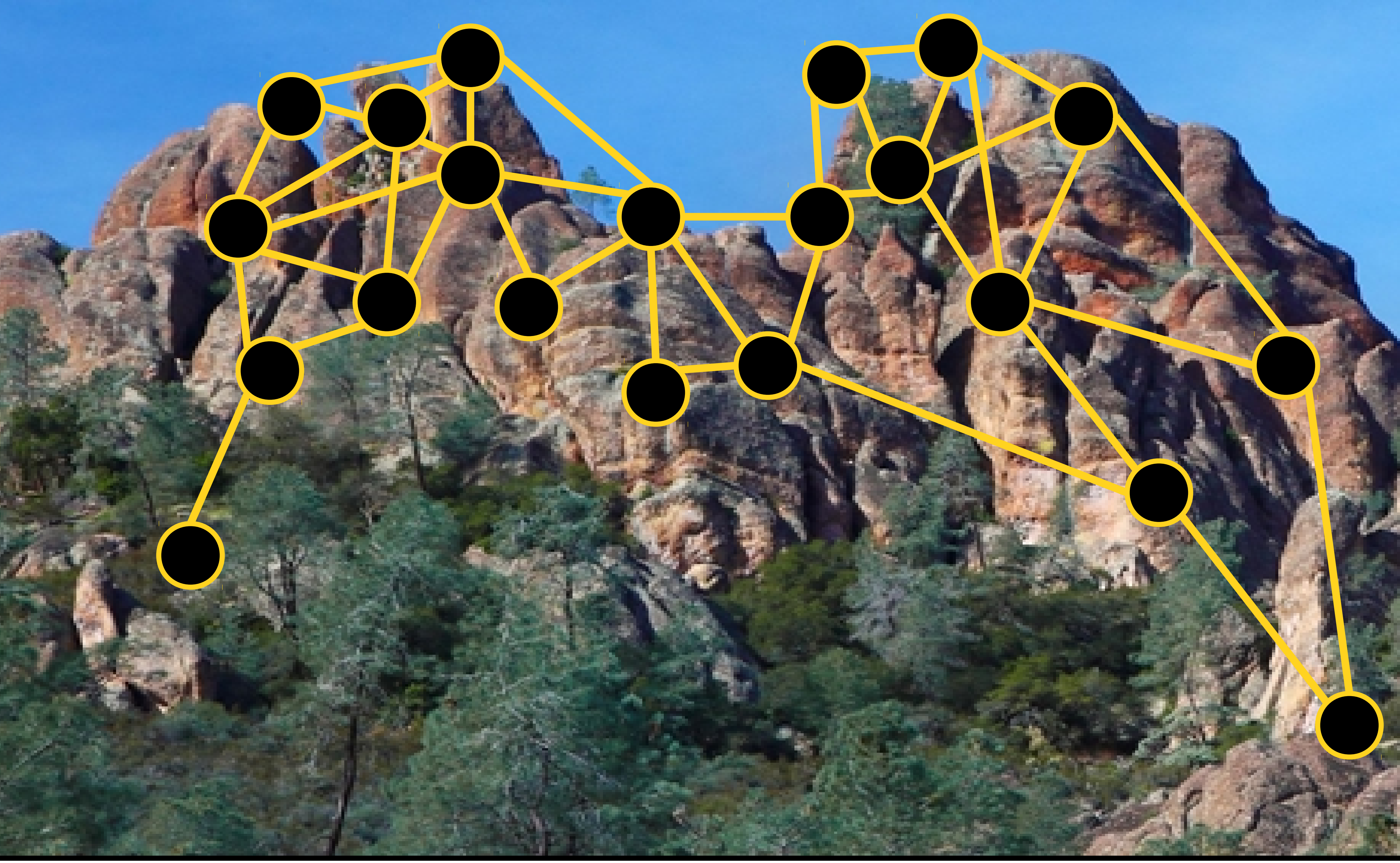

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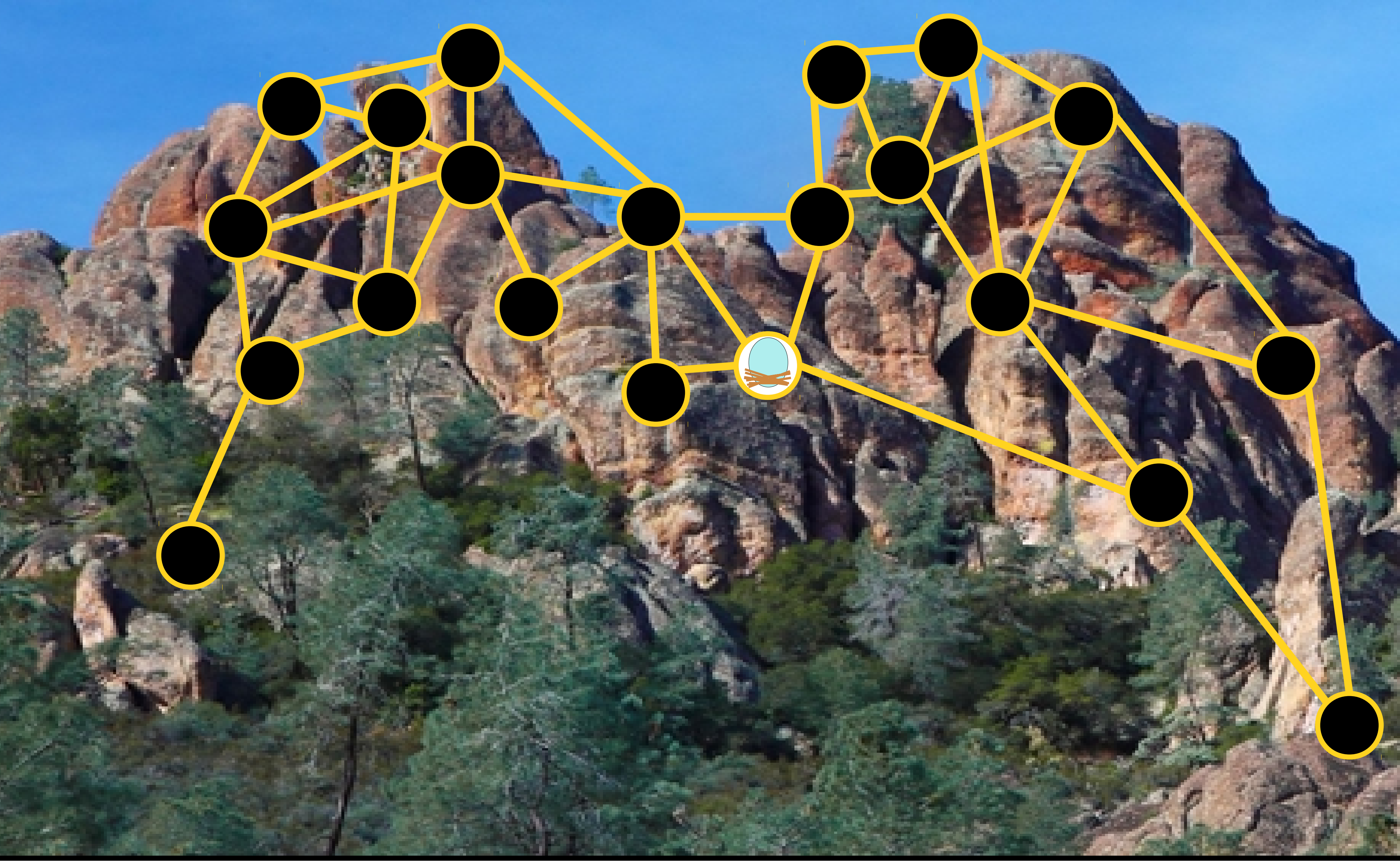

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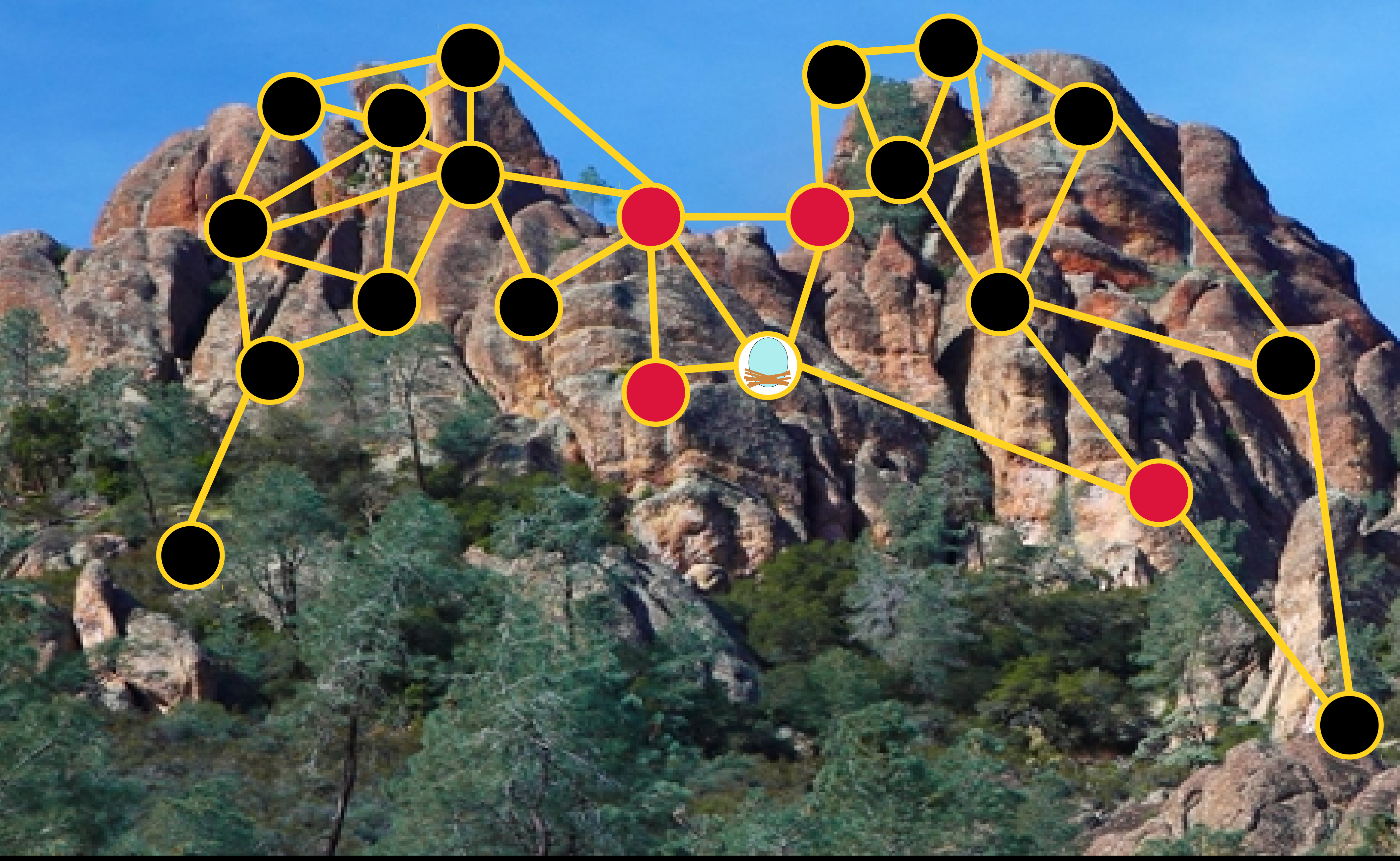

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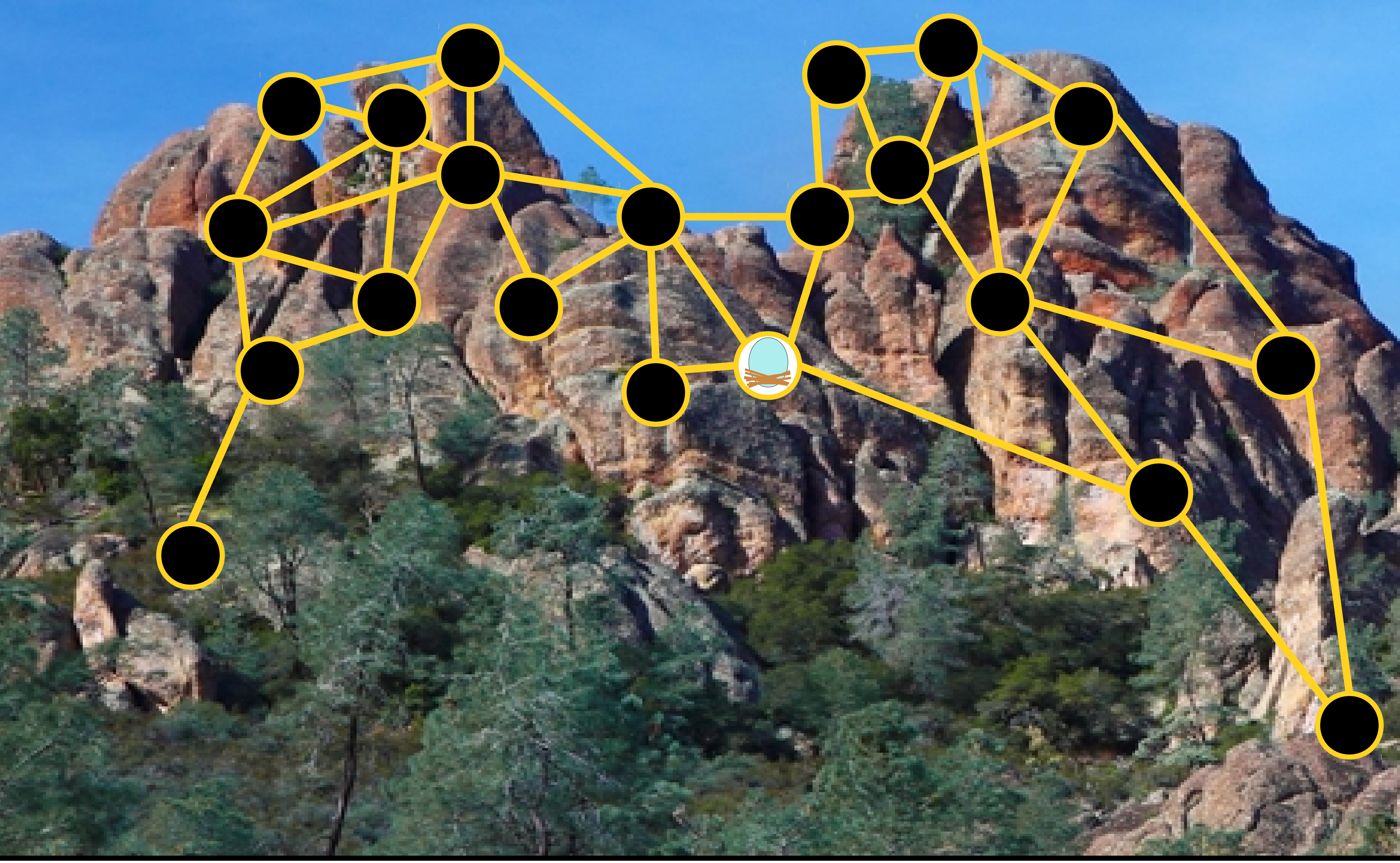

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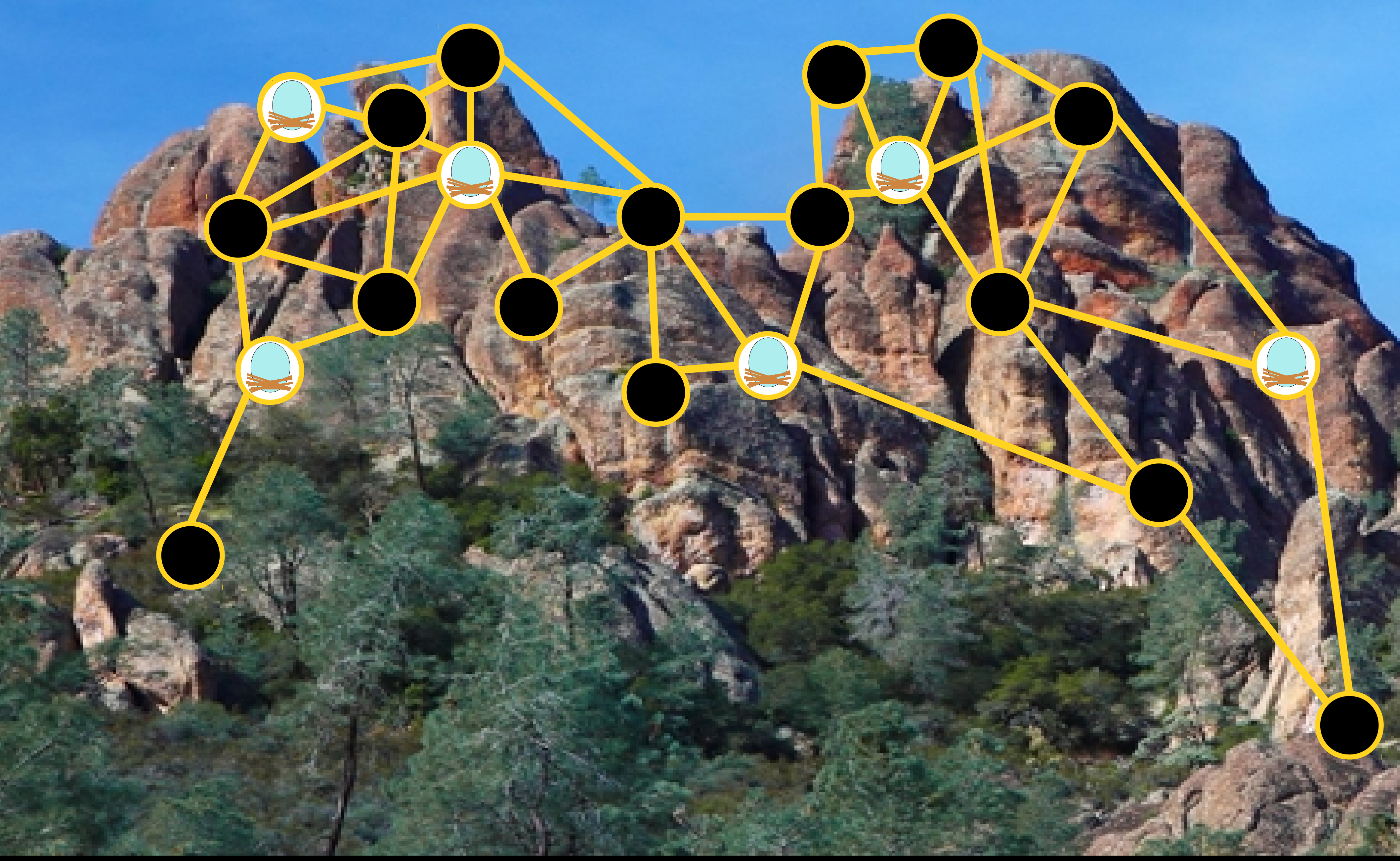

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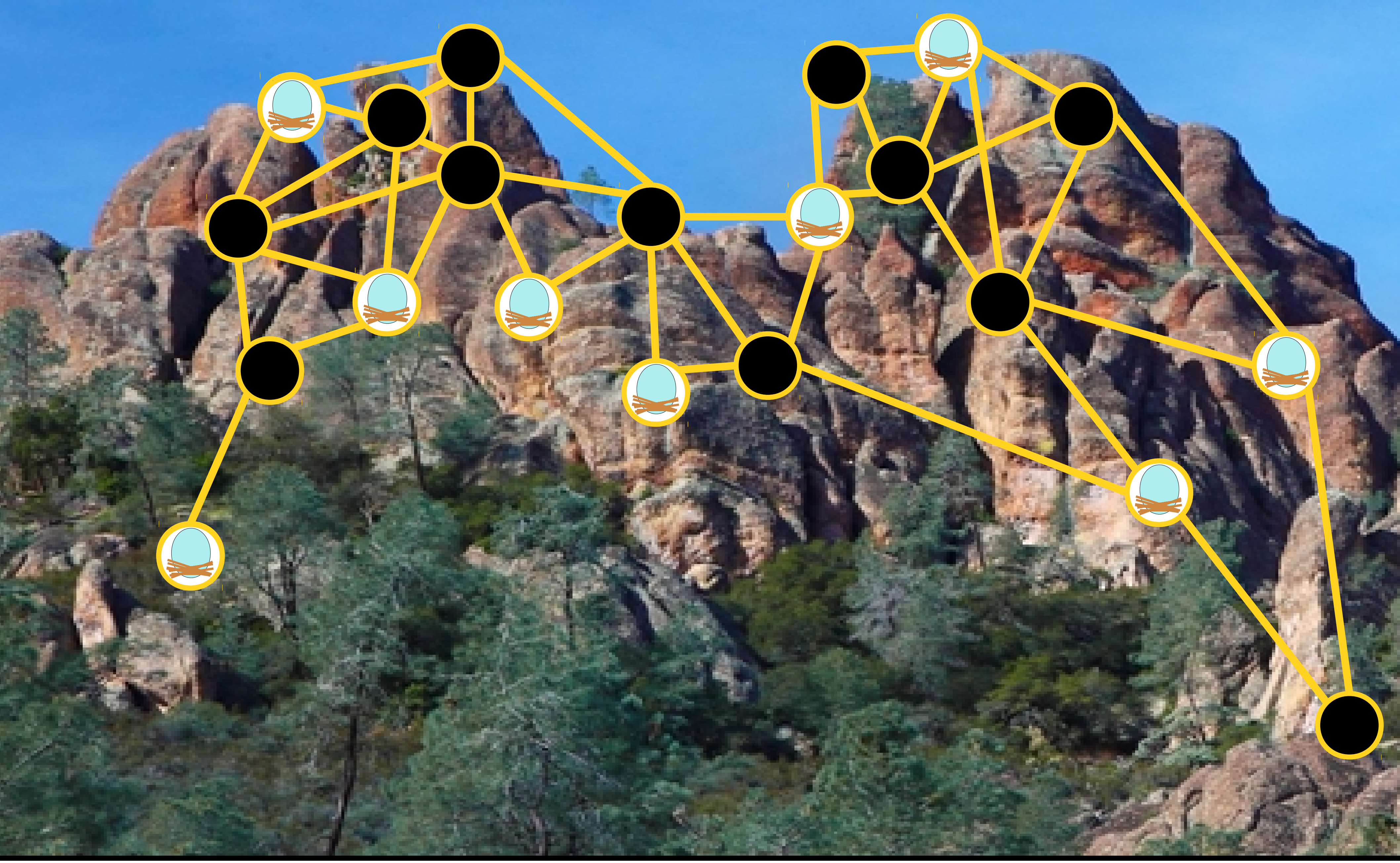

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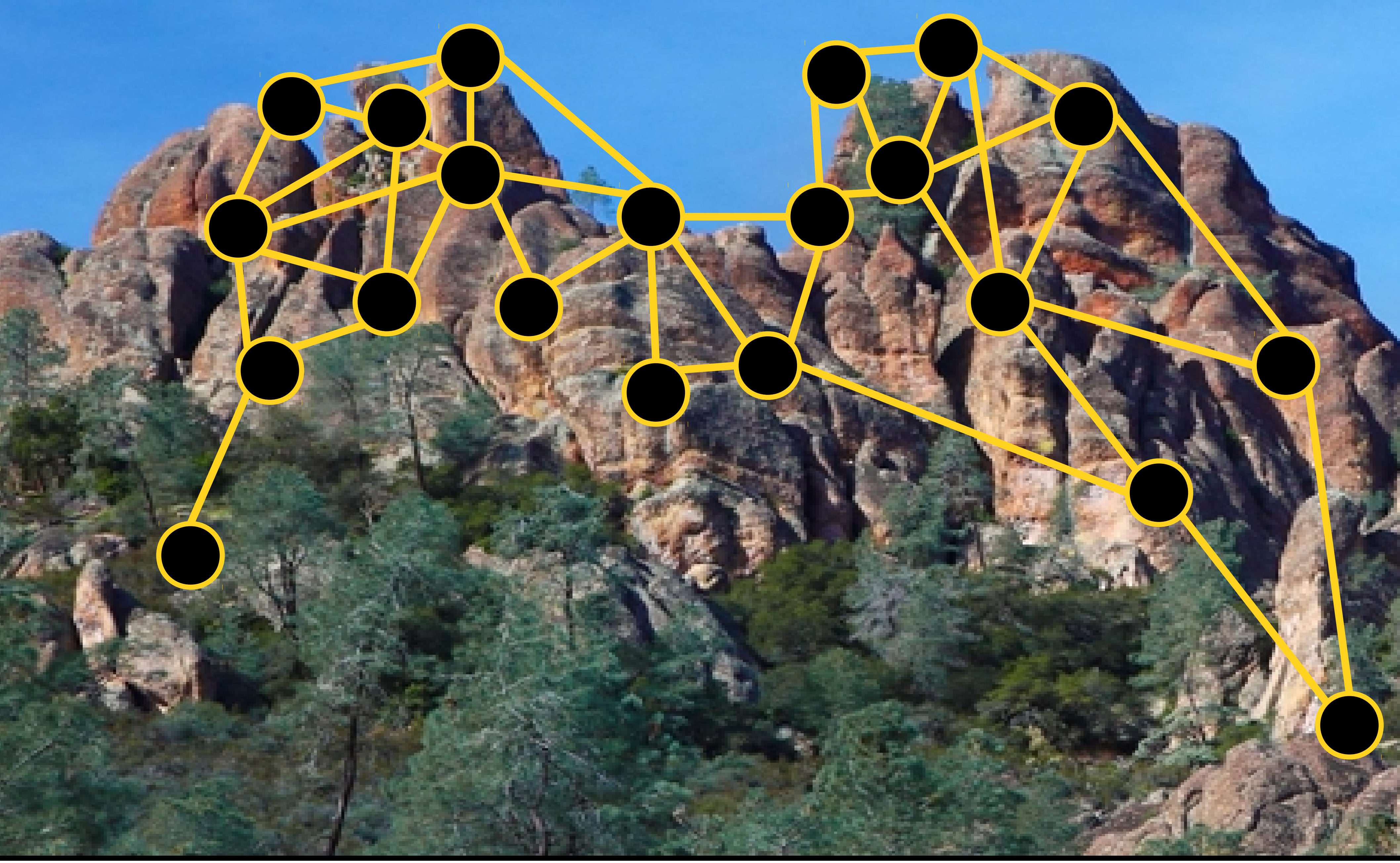

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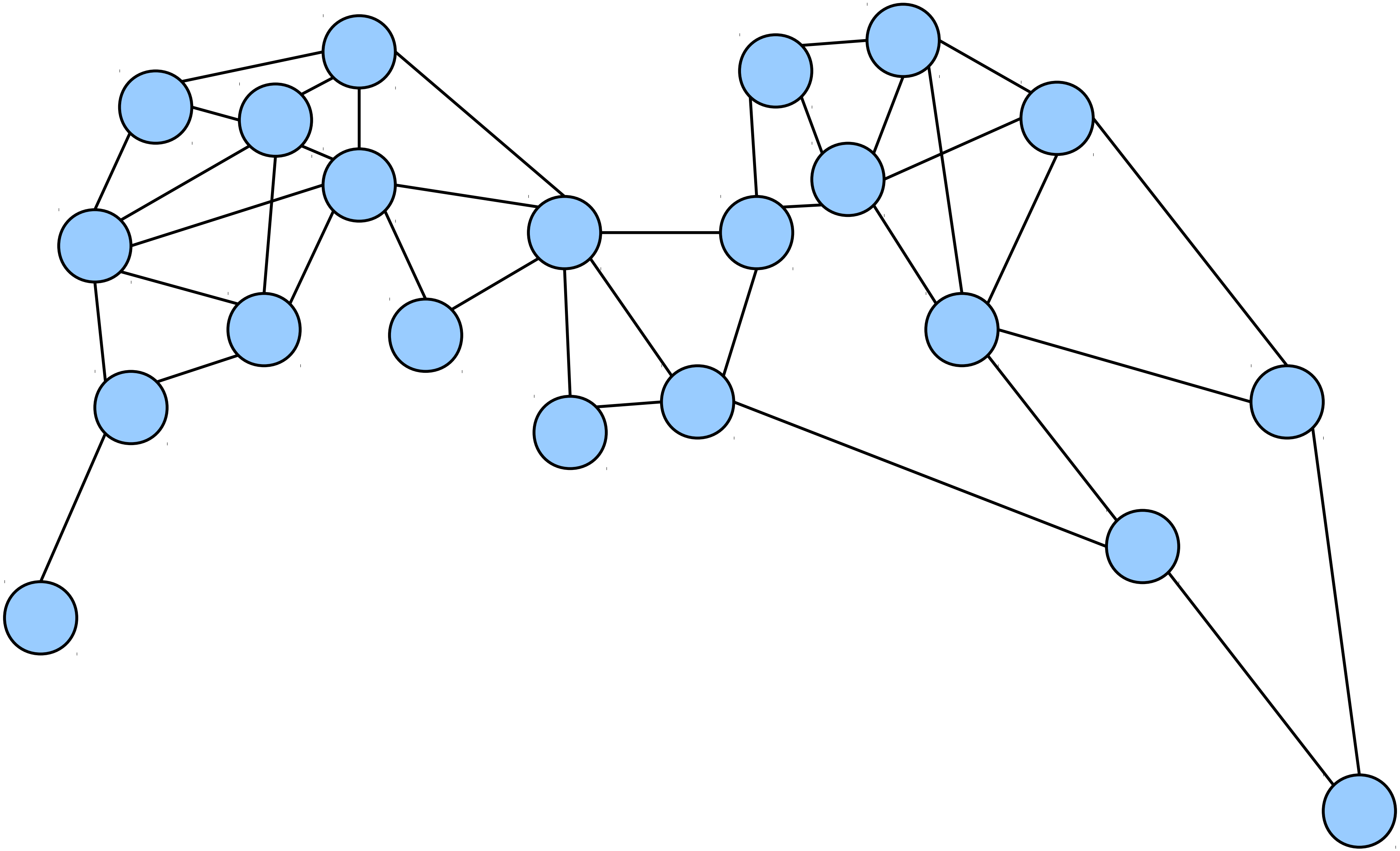

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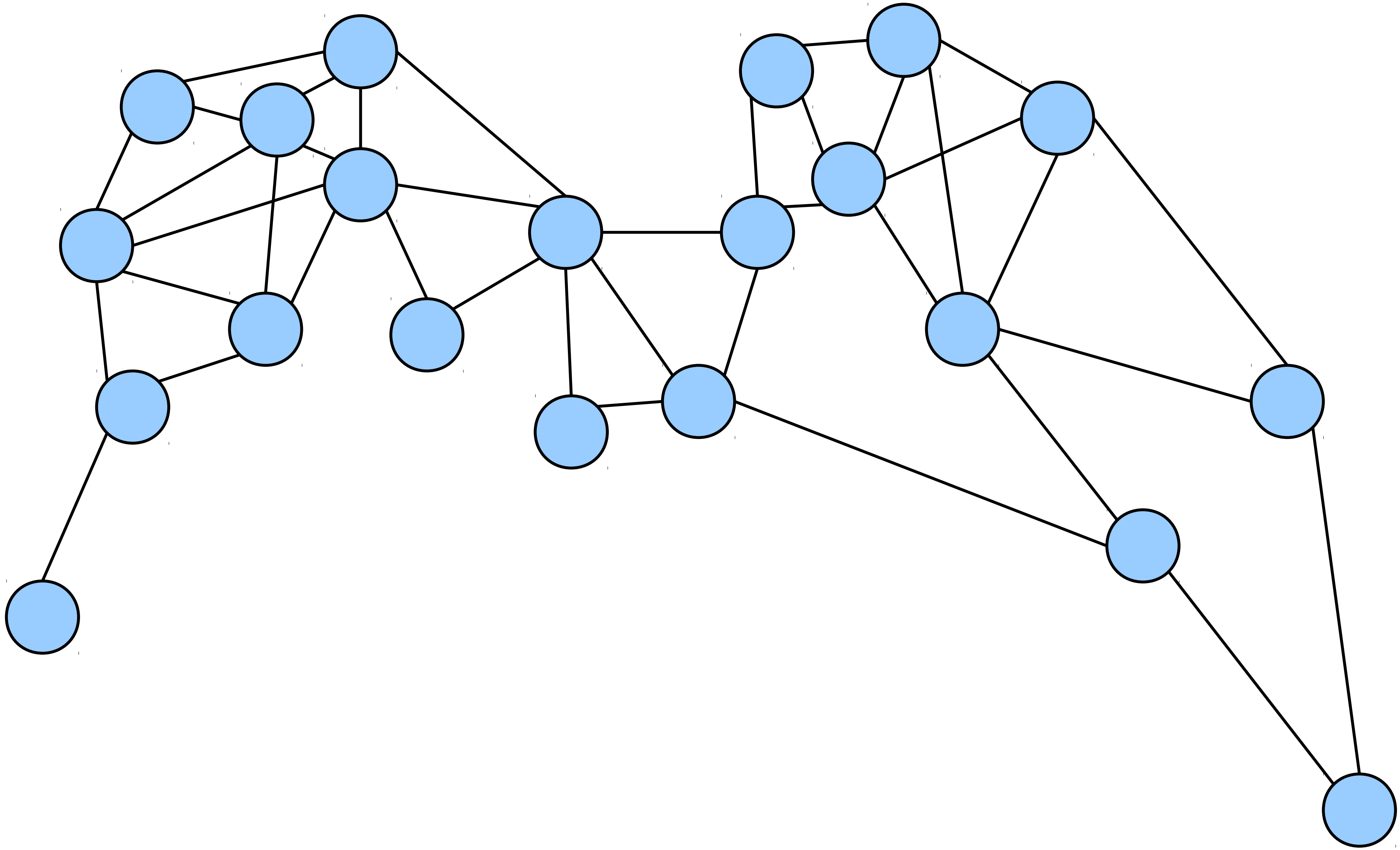


Choose a set of nodes, no two of which are adjacent.

Independent Sets

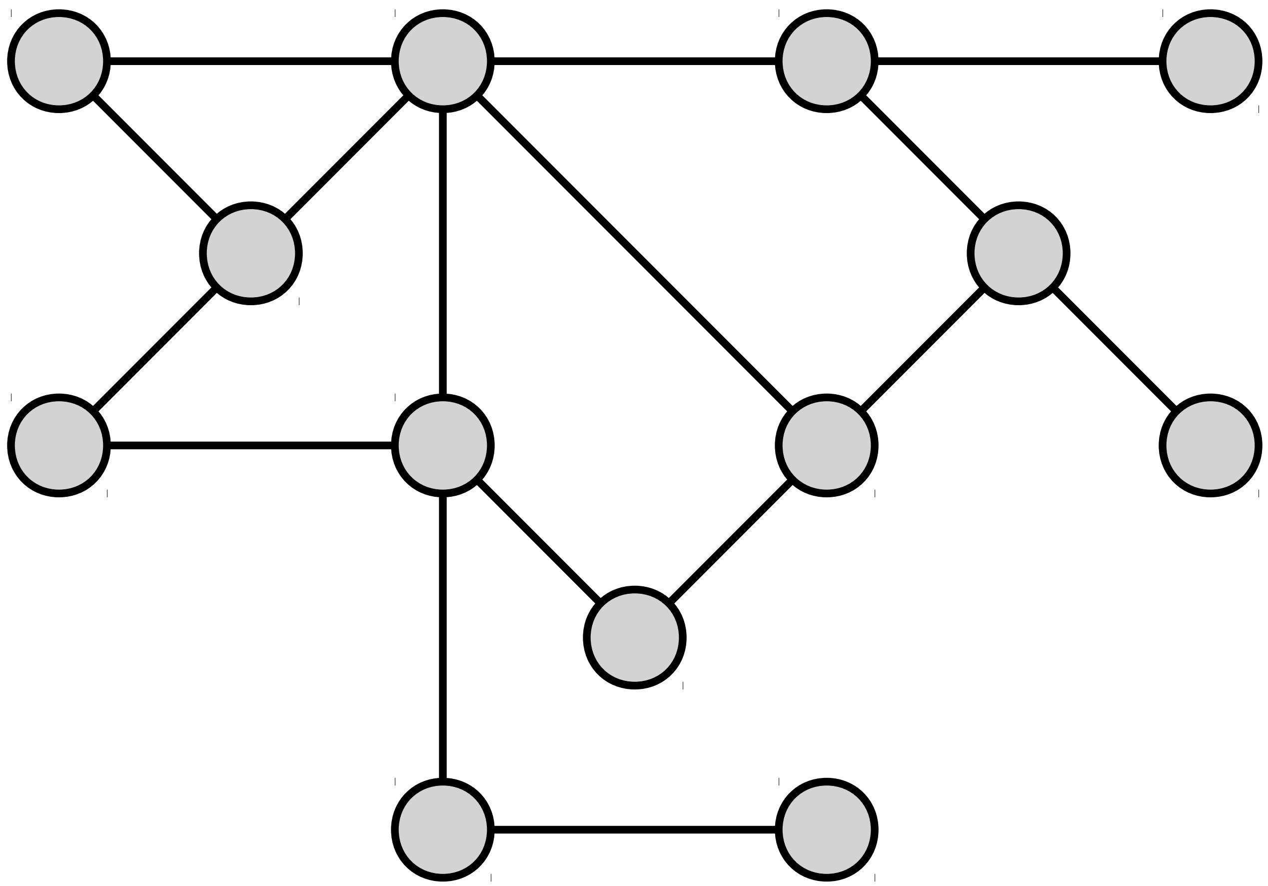
- If $G = (V, E)$ is an (undirected) graph, then an ***independent set*** in G is a set $I \subseteq V$ such that

$$\forall u \in I.\ \forall v \in I.\ \{u, v\} \notin E.$$

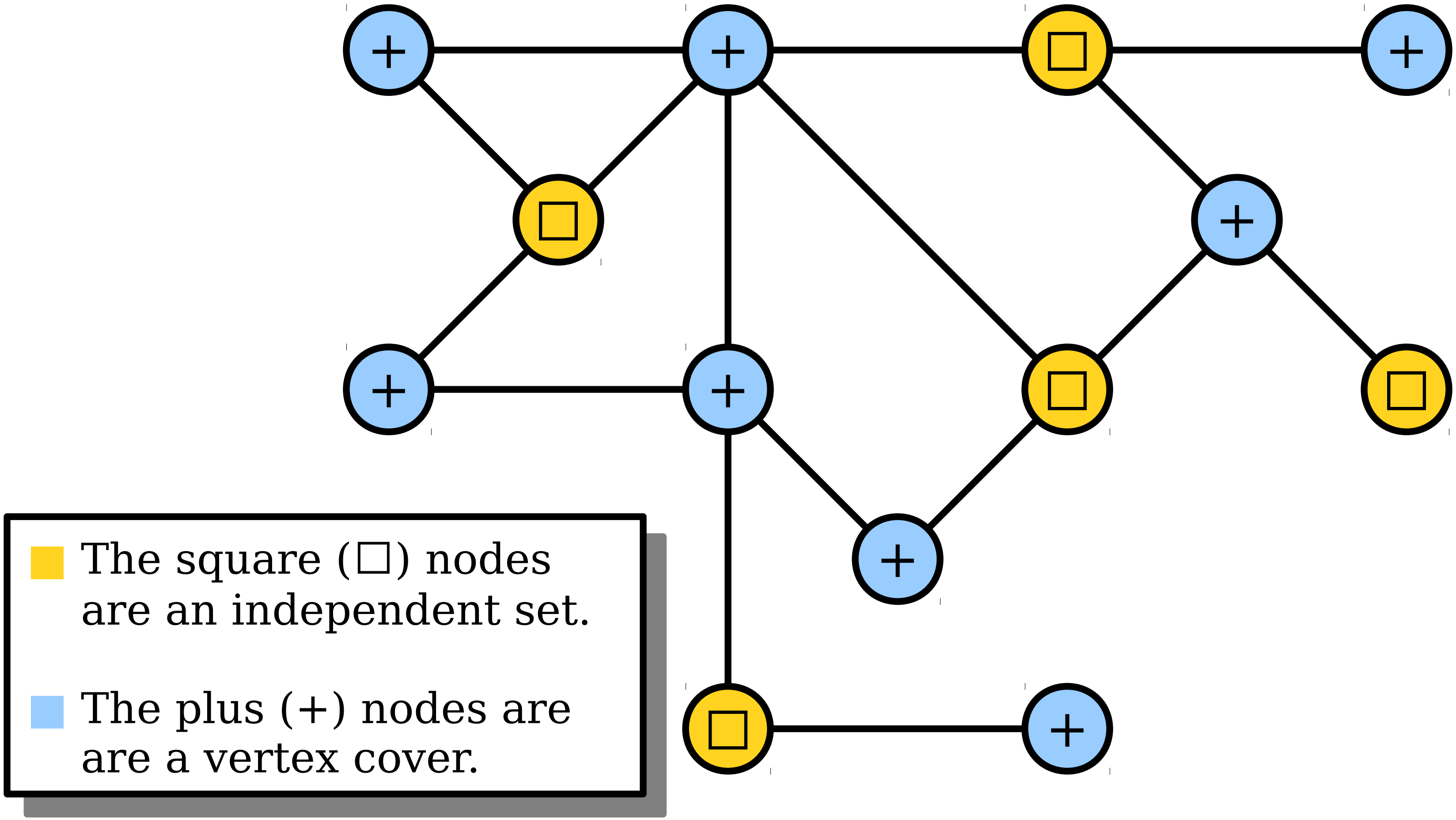
("No two nodes in I are adjacent.")

- Independent sets have applications to resource optimization, conflict minimization, error-correcting codes, cryptography, and more.

A Connection



Independent sets and vertex covers are related.



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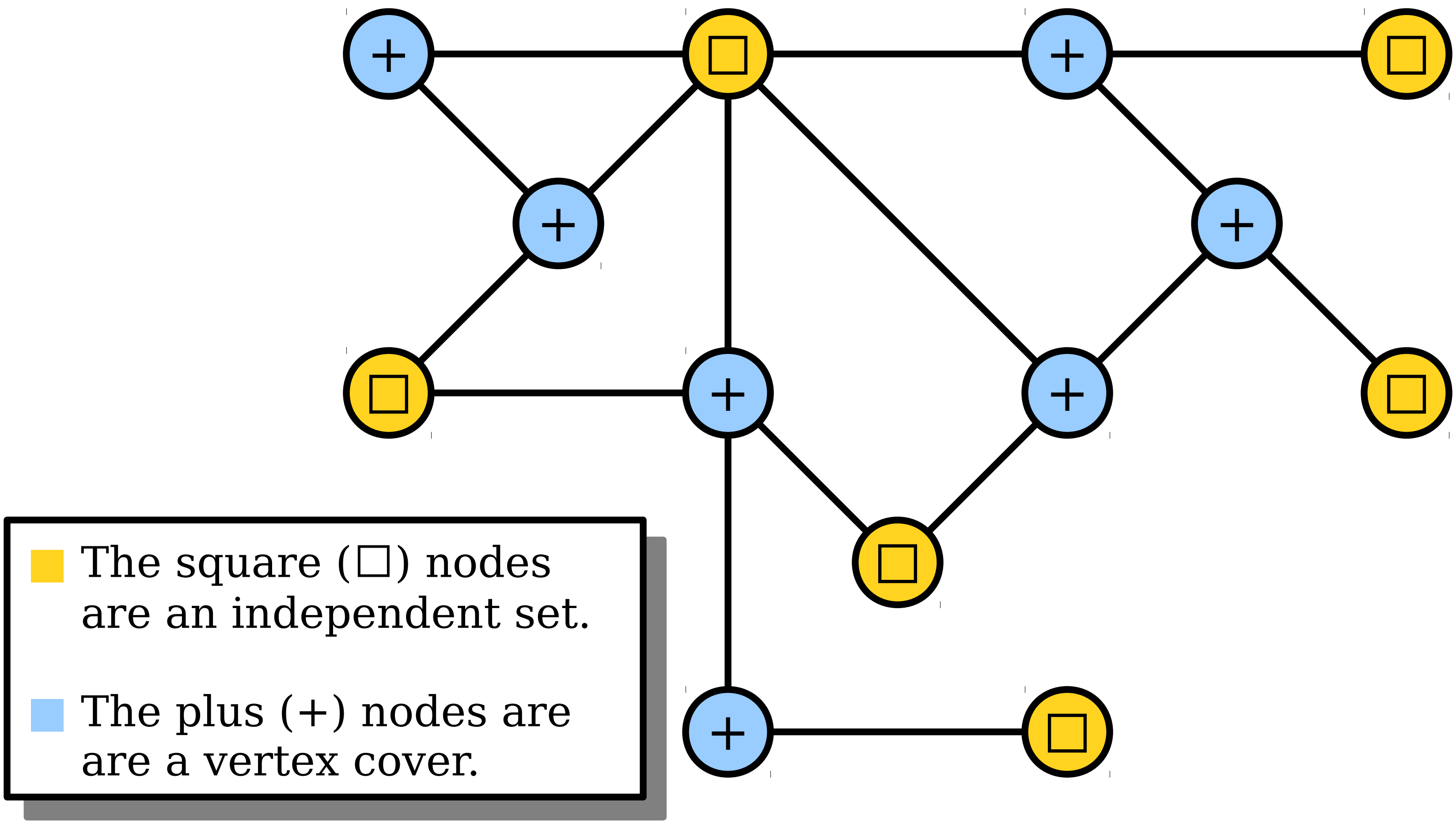

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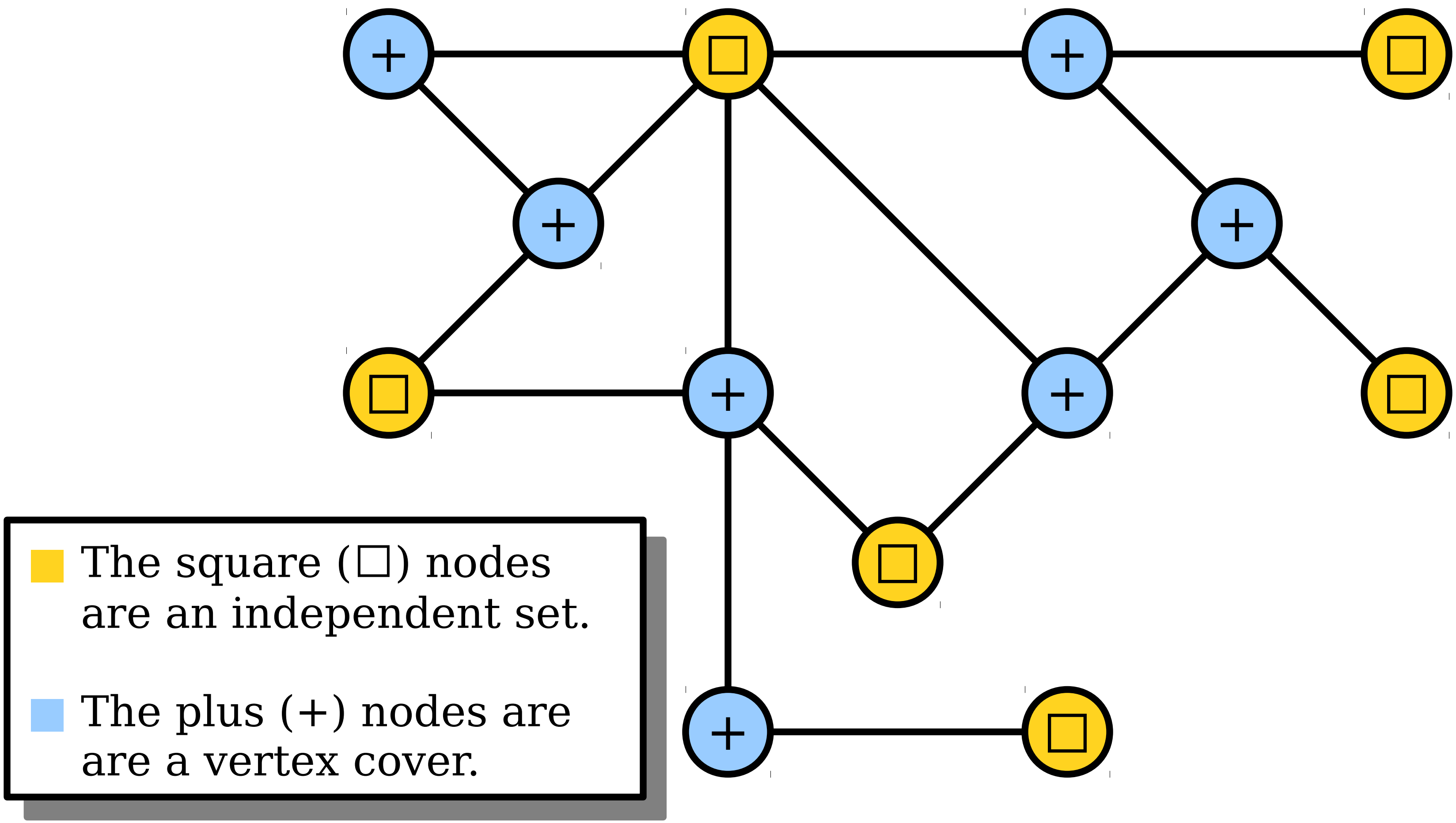

Theorem: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. Then C is a vertex cover of G if and only if $V - C$ is an independent set in G.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set in G.

What We're Assuming	***What We Need To Show***
G is a graph.	$V - C$ is an independent set in G.
C is a vertex cover of G.	$\forall x \in V - C.$
$\forall u \in V. \forall v \in V. (\{u, v\} \in E \to$	$\forall y \in V - C.$
$u \in C \lor v \in C$	$\{x, y\} \notin E.$
$)$	

Based on the assume/prove columns here, which of u, v, x, and y should we introduce?

Answer at
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$)$	

We're assuming a universally-quantified statement. That means we *don't do anything right now* and instead wait for an edge to present itself.

We need to prove a universally-quantified statement. We'll ask the reader to pick arbitrary choices of x and y for us to work with.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set in G.

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$x \in V$ and $x \notin C$.	$\{x, y\} \notin E.$
$y \in V$ and $y \notin C$.	

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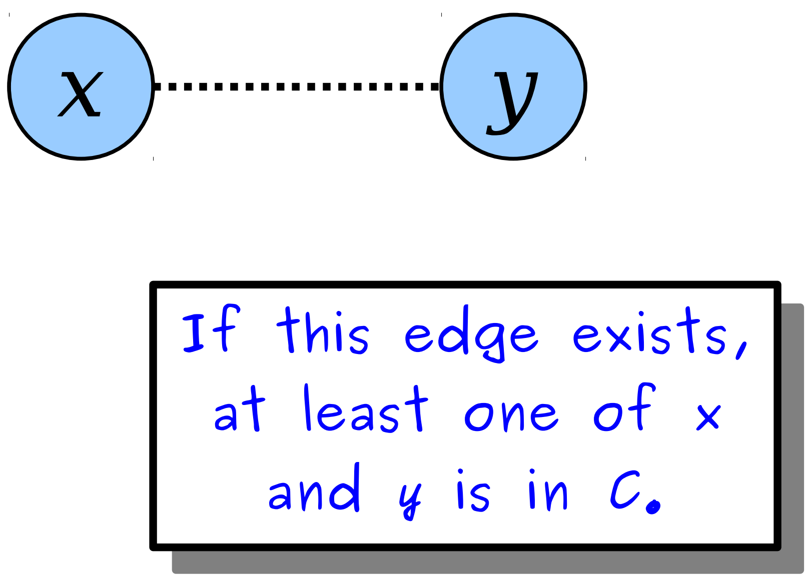

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set in G.

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G is a graph.	$V - C$ is an independent set in G.
C is a vertex cover of G.	$\forall x \in V - C.$
$\forall u \in V. \forall v \in V. (\{u, v\} \in E \to$	$\forall y \in V - C.$
$u \in C \quad \lor \quad v \in C$	$\{x, y\} \notin E.$
$)$	
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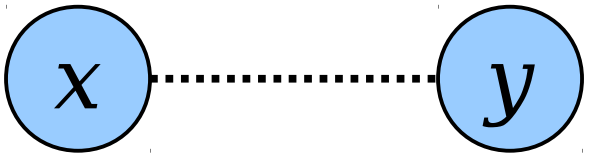

If this edge exists, at least one of x and y is in C.

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Proof:

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Proof: Assume C is a vertex cover of G.

> There's no need to introduce G or C here. That's done in the statement of the lemma itself.

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Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V – C$ is an independent set of G.

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Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V – C$, we know that $x \notin C$ and $y \notin C$. However, since C is a vertex cover of G and $\{x, y\} \in E$, we also see that $x \in C$ or $y \in C$, contradicting our previous statement.

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Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set in G.

Taking Negations

- What is the negation of this statement, which says "*C* is a vertex cover?"

$$\forall u \in V.\ \forall v \in V.\ (\{u, v\} \in E \to$$
$$u \in C \quad \lor \quad v \in C$$
$$)$$

Taking Negations

- What is the negation of this statement, which says "C is a vertex cover?"

$$\neg\forall u \in V.\ \forall v \in V.\ (\{u, v\} \in E \rightarrow$$
$$u \in C \quad \lor \quad v \in C$$
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Taking Negations

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$$\exists u \in V.\ \exists v \in V.\ \neg(\{u, v\} \in E \to$$
$$u \in C \quad \lor \quad v \in C$$
$$)$$

Taking Negations

- What is the negation of this statement, which says “C is a vertex cover?”

$$\exists u \in V.\ \exists v \in V.\ (\{u, v\} \in E\ \land$$
$$\lnot(u \in C \quad \lor \quad v \in C)$$
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Taking Negations

- What is the negation of this statement, which says "C is a vertex cover?"

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$$
\begin{aligned}
&\exists u \in V.\ \exists v \in V.\ (\{u, v\} \in E\ \land \\
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&)
\end{aligned}
$$

- This says "there is an edge where both endpoints aren't in C."

Taking Negations

- What is the negation of this statement, which says "*V* – *C* is an independent set?"

How do we express the following statement in FOL?

"*V* - *C* is not an independent set."

Answer at
https://pollev.com/cs103

Taking Negations

- What is the negation of this statement, which says "*V* – *C* is an independent set?"

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- What is the negation of this statement, which says "$V - C$ is an independent set?"

$$\exists u \in V - C.\ \exists v \in V - C.\ \{u, v\} \in E$$

- This says "there are two adjacent nodes in $V - C$."

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set in G.

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$\exists u \in V. \exists v \in V. (\{u, v\} \in E \land$ $u \notin C \quad \land \quad v \notin C$ $)$	$\exists y \in V - C.$ $\{x, y\} \in E.$

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We're assuming an existentially-quantified statement, so we'll *immediately* introduce variables u and v.

We're proving an existentially-quantified statement, so we *don't* introduce variables x and y. We're on a scavenger hunt!

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Any ideas about what we should pick x and y to be?

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Since C is not a vertex cover of G, we know that there exists nodes $x, y \in V$ where $\{x, y\} \in E$, where $x \notin C$, and where $y \notin C$.

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Since C is not a vertex cover of G, we know that there exists nodes $x, y \in V$ where $\{x, y\} \in E$, where $x \notin C$, and where $y \notin C$. Because $x \in V$ and $x \notin C$, we know that $x \in V – C$.

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This means that $\{x, y\} \in E$, that $x \in V - C$, and that $y \in V - C$, and therefore that $V - C$ is not an independent set of G, as required.

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Finding an IS or VC

- The previous theorem means that finding a large IS in a graph is equivalent to finding a small VC.
  - If you've found one, you've found the other!
- ***Open Problem:*** Design an algorithm that, given an n-node graph, finds either the largest IS or smallest VC "efficiently," where "efficiently" means "in time $O(n^k)$ for some $k \in \mathbb{N}$."
  - There's a $1,000,000 bounty on this problem – we'll see why in Week 10.

Recap for Today

- A ***graph*** is a structure for representing items that may be linked together. ***Digraphs*** represent that same idea, but with a directionality on the links.
- Graphs can't have ***self-loops***; digraphs can.
- ***Vertex covers*** and ***independent sets*** are useful tools for modeling problems with graphs.
- The complement of a vertex cover is an independent set, and vice-versa.

Next Time

- ***Paths and Trails***
  - Walking from one point to another.
- ***Graph Complements***
  - Looking at what's missing.
- ***Indegrees and Outdegrees***
  - Counting how many neighbors you have, in the directed case.
- ***Teleporting a Train***
  - Can you get stuck in a loop?